Asymptotic enumeration of graphical representations

Yunsong Gan

joint work with Pablo Spiga and Binzhou Xia

Southern University of Science and Technology

September 2025

DRR/GRR

Let G be a finite group and $S \subseteq G$. A Cayley digraph $\mathrm{Cay}(G, S)$ of G is a digraph with vertex set G such that (g, h) is an arc if and only if $hg^{-1} \in S$. If $S = S^{-1}$, then $\mathrm{Cay}(G, S)$ is a Cayley graph of G.

We call a Cayley (di)graph of G a (di)graphical regular representation, or GRR (DRR) for short, if its automorphism group is isomorphic to G.

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A natural problem is to determine which finite groups admit a DRR or GRR.

Existence of DRRs/GRRs

Theorem (Babai,1980)

C_2^2, C_2^3, C_2^4, C_3^2 and Q_8 are the only five groups without DRRs.

L. Babai, Finite digraphs with given regular automorphism groups, *Period. Math. Hung.* 11 (1980), 257–270.

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Theorem (Godsil,1981)

Apart from abelian groups of exponent greater than 2, generalized dicyclic groups and 13 small solvable groups, every finite group admits a GRR.

C. D. Godsil, On the full automorphism group of a graph, *Combinatorica* 1 (1981), 243–256.

Asymptotic enumeration of DRRs

Babai and Godsil conjectured in early 1980s that almost all finite Cayley digraphs are DRRs, namely,

$$\frac{|\{S : \mathrm{Cay}(G, S) \text{ is a DRR}\}|}{|S|} \to 1 \ \text{ as } \ |G| \to \infty.$$

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Theorem (Morris-Spiga,2021)

Let G be a finite group of order n. When n is sufficiently large, the proportion of subsets S of G such that $\mathrm{Cay}(G, S)$ is not a DRR is at most $2^{-\frac{bn^{0.499}}{4\log_2^3 n}+2}$, where b is an absolute constant.

J. Morris and P. Spiga, Asymptotic enumeration of Cayley digraphs, *Israel J. Math.* 242 (2021), 401–459.

Asymptotic enumeration of GRRs

Babai, Godsil, Imrich and Lovász conjectured that, except for the groups G that are abelian of exponent greater than 2 or generalised dicyclic, almost all finite Cayley graphs of G are GRRs, namely,

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Theorem (Xia-Zheng,2023)

Let G be a finite group of order n such that G is neither abelian of exponent greater than 2 nor generalized dicyclic. When n is sufficiently large, the proportion of inverse-closed subsets S of G such that $\mathrm{Cay}(G, S)$ is not a GRR is at most $2^{-\frac{n^{0.499}}{8\log_2^3 n}+\log_2^2 n+3}$.

B. Xia and S. Zheng, Asymptotic enumeration of graphical regular representations, *Proc. Lond. Math. Soc.*(3) 127 (2023), 1424–1450.

m-Cayley (di)graphs

Let m be a positive integer. Let

$$\mathcal{S} = (S_{i,j})_{m\times m} = \begin{pmatrix} S_{1,1} & S_{1,2} & \cdots & S_{1,m} \\ S_{2,1} & S_{2,2} & \cdots & S_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ S_{m,1} & S_{m,2} & \cdots & S_{m,m} \end{pmatrix}$$

be a set-matrix of G. The m-Cayley digraph $\mathrm{Cay}(G, \mathcal{S})$ of G with respect to $\mathcal{S}$ is the digraph with vertex set $G \times \{1, \dots, m\}$ and arc set

$$\bigcup_{i,j\in\{1,\dots,m\}} \left\{ ((g,i),(sg,j)) \,\middle|\, s \in S_{i,j},\ g \in G \right\}.$$

We call a set-matrix $\mathcal{S}$ inverse-closed if $S_{j,i} = S_{i,j}^{-1}$ for all $i, j \in \{1, \dots, m\}$. For such $\mathcal{S}$, we call the digraph $\mathrm{Cay}(G, \mathcal{S})$ an m-Cayley graph as it is undirected.

Existence of DmSRs/GmSRs

An m-Cayley (di)graph of a finite group G is called a (di)graphical m-semiregular representation, abbreviated as GmSR (DmSR), if its automorphism group is isomorphic to G.

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Theorem(Du, Feng, Spiga,2021)

Every group of order greater than 8 has a GmSR and DmSR for all $m \geq 2$.

J.-L. Du, Y.-Q. Feng and P. Spiga, A classification of the graphical m-semiregular representation of finite groups, *J. Combin. Theory Ser. A* 171 (2020).

Asymptotic enumeration of DmSRs/GmSRs

For a finite group G of sufficiently large order, are almost all m-Cayley (di)graphs of G GmSRs (DmSRs)? That is,

$$\frac{|\{\mathcal{S} : (S = S^{-1},)\mathrm{Cay}(G, S) \text{ is a D}m\text{SR}(\text{G}m\text{SR})\}|}{|\{\mathcal{S} : (\mathcal{S} = \mathcal{S}^{-1})\}|} \to 1 \ \text{ as } \ |G| \to \infty.$$

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Theorem 1(Gan, Spiga, Xia, 2026)

Fix an integer $m \geq 2$, and let G be a finite group of order n. When n is sufficiently large, the proportion of (inverse-closed) set-matrices $\mathcal{S}$ of G such that $\mathrm{Cay}(G, \mathcal{S})$ is a DmSR (GmSR) is greater than $1 - m^2/\sqrt{n}$.

Asymptotic enumeration of DmSRs/GmSRs

Strategy to prove Theorem 1: estimate each size of sets

$\mathcal{Z}_1 = \{\mathcal{S} : \exists i \in \{1, \ldots, m\}$ s.t. $\mathrm{Aut}(\mathrm{Cay}(G, \mathcal{S}))$ does not stabilize $G_i\}$,

$\mathcal{Z}_2 = \{\mathcal{S} \notin \mathcal{Z}_1 : \exists i \in \{1, \ldots, m\}$ s.t. $A_i \ncong G\}$, and

$\mathcal{Z}_3 = \{\mathcal{S} \notin \mathcal{Z}_1 \cup \mathcal{Z}_2 : \mathrm{Cay}(G, \mathcal{S})$ is not a DmSR/GmSR$\}$.

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$$\mathcal{Z}_3 = \{\mathcal{S} \notin \mathcal{Z}_1 \cup \mathcal{Z}_2 : \mathrm{Cay}(G, \mathcal{S}) \text{ is not a D}m\text{SR/G}m\text{SR}\}.$$

The validity of Theorem 1 relies significantly on the presence of arcs within each G-orbit in most m-Cayley (di)graphs of G. Establishing an analog of Theorem 1 becomes notably more challenging, even for $m = 2$, if arcs within each G-orbit are prohibited.

Haar graphs

Let G be a finite group and let $S \subseteq G$. Then a Haar graph $\mathrm{H}(G, S)$ is a graph with vertex set $G_+ \cup G_-$ such that g_+ and h_- are adjacent if and only if $hg^{-1} \in S$.

When G is abelian, there exists an automorphism ι of order two such that the group $G \rtimes \langle \iota \rangle$ acts regularly on the vertex set. Hence no Haar graph of an abelian group is an HGR.

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Theorem(Morris-Spiga, 2025)

They have classified finite groups admitting an HGR.

J. Morris and P. Spiga, Haar graphical representations of finite groups and an application to poset representations, *J. Combin. Theory Ser. B* 173 (2025), 279–304.

Asymptotic enumeration of HGRs

Theorem 2(Gan, Spiga, Xia, 2026)

Let $\varepsilon \in (0, 0.1]$, and let n_ε be a positive integer such that for all $n \geq n_\varepsilon$,

$$(6 + 2\log_2 n)(n^{0.5-\varepsilon} + \log_2 n)^2 + (1 - \log_2 n)(n^{0.5-\varepsilon} + \log_2 n) + \log_2^2 n + 2\log_2 n < n^{1-\varepsilon}.$$

Let G be a finite group of order n, and let
$f_\varepsilon(n) = \frac{n^{0.5-\varepsilon}}{24(\log_2 n)^{2.5}} - \frac{3\log_2^2 n}{4} - 15.$

(a) If G is nonabelian, then the number of subsets S of G such that $\mathrm{H}(G, S)$ is not an HGR is less than $2^{n - f_\varepsilon(n)}$.

(b) If G is abelian, then the number of subsets S of G such that the automorphism group of $\mathrm{H}(G, S)$ is not isomorphic to $G \rtimes \langle \iota \rangle$ is less than $2^{n - f_\varepsilon(n) - 1}$.

Asymptotic enumeration of HGRs

Let G be a finite group. We need to enumerate the size of the set

$$\mathcal{X} = \{S \subset G : \mathrm{Aut}(\mathrm{H}(G, S)) > G\}.$$

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Lemma

Let X be a finite group, and let G be a nonabelian subgroup of index 2 in X. Then $|\mathcal{I}(X \setminus G)| \leq 3|G|/4$.

The key is to enumerate the size of the set

$$\mathcal{Y} = \{S \subset G : \mathrm{Aut}^{+}(\mathrm{H}(G, S)) > G\}.$$

Asymptotic enumeration of HGRs

Let $S \in \mathcal{Y}$. Then $\mathrm{Aut}^+(\mathrm{H}(G, S)) > G$, and there exists a group $M \leq \mathrm{Aut}^+(\mathrm{H}(G, S))$ such that G is maximal in M. Moreover, there is an exact factorization

$$M = M_x G = M_y G,$$

for each pair of vertices $x \in G_+$ and $y \in G_-$.

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for each pair of vertices $x \in G_+$ and $y \in G_-$.

By three reduction steps (including Babai-Godsil-like reduction and Morris-Spiga reduction), it suffices to count the number of $S \in \mathcal{Y}$ for which there exists an M satisfying

- $\mathrm{Core}_M(G) = 1$, and
- $|M_x| > 2^{|G|^{0.5-\epsilon}}$.

Count the number of primitive groups with a large regular subgroup.

Primitive groups with a large regular subgroup

HS/HC type

Let M be a primitive group of HS or HC type with stabilizer G such that $|G| > 132$. Then M has no regular subgroup with order greater than $2^{|G|^{0.4}}$.

AS type

Let G be finite group of order $n \geq 2^{11}$. Then $|\mathcal{Z}_{\mathsf{AS}}(G)| < 2^{3(\log_2 n)+75}$.

PA type

Let G be a finite group with $|G| \geq 2^{57}$. Then $\mathcal{Z}_{\mathsf{PA}}(G) = \emptyset$.

Primitive groups with a large regular subgroup

CD type

Let G be a finite group of order n. Then

$$|\mathcal{Z}_{\mathrm{CD}}(G)| < 2^{\frac{3}{4}n+2\log_2^4 n+\log_2^3 n+1702\log_2^2 n+2\log_2 n}.$$

HA/SD/TW type

Let G be a finite group of order $n \geq 2^{23}$, and let $\varepsilon \in (0, 0.1]$. Then

$$|\mathcal{Z}_{\mathrm{HA}}(G,\varepsilon) \cup \mathcal{Z}_{\mathrm{SD}}(G,\varepsilon) \cup \mathcal{Z}_{\mathrm{TW}}(G,\varepsilon)| < 2^{n-\frac{n^{0.5-\varepsilon}}{8\log_2^2 n}+\frac{\log_2^2 n}{2}+7}.$$

Asymptotic enumeration of HGRs

Proposition

Let $\varepsilon \in (0, 0.1]$, and let n_ε be a positive integer such that the inequality of Theorem 2 holds for all $n \geq n_\varepsilon$. Let G be a finite group of order n. Then the number of subsets S of G such that $\mathrm{Aut}^+(\mathrm{H}(G,S)) > G$ is less than $2^{n - \frac{n^{0.5-\varepsilon}}{24(\log_2 n)^{2.5}} + \frac{3\log_2^2 n}{4} + 15}$.

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Theorem 3

Let $\varepsilon \in (0, 0.1]$, and let n_ε be a positive integer such that the inequality of Theorem 2 holds for all $n \geq n_\varepsilon$. Let G be a finite group of order n, and let $h_\varepsilon(n) = \frac{n^{0.5-\varepsilon}}{24(\log_2 n)^{2.5}} - \log_2^2(2n) - \frac{3\log_2^2 n}{4} - 2\log_2 n - 15$.

(a) If G is nonabelian, then the proportion of HGRs of G among Haar graphs of G, up to isomorphism, is greater than $1 - 2^{-h_\varepsilon(n)}$.

(b) If G is abelian, then the proportion of Haar graphs of G whose automorphism group is isomorphic to $G \rtimes \langle \iota \rangle$, among Haar graphs of G, up to isomorphism, is greater than $1 - 2^{-h_\varepsilon(n)}$.

Asymptotic enumeration of m-PGSRs

A set-matrix $\mathcal{S}$ of G is said to be skew, if it is inverse-closed and satisfies $S_{i,i} = \emptyset$ for each $i \in \{1, \ldots, m\}$. An m-Cayley graph $\mathrm{Cay}(G, \mathcal{S})$ with skew $\mathcal{S}$ is called an *m-partite graphical semiregular representation* (m-PGSR) of G, if its automorphism group is isomorphic to G.

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Theorem 4

Fix an integer $m \geq 3$, and let G be a finite group of order n. When n is sufficiently large, the proportion of skew set-matrices $\mathcal{S}$ of G such that $\mathrm{Cay}(G, \mathcal{S})$ is an m-PGSR is greater than $1 - m^2/\sqrt{n}$.

Mckay-Praeger Conjecture

McKay-Praeger conjectured that almost all finite vertex-transitive (di)graphs are Cayley, namely,

$$\frac{|\{\text{a Cayley (di)graph of order } n\}|}{|\{\text{a vertex-transitive (di)graph of order } n\}|} \to 1 \quad \text{as} \quad n \to \infty.$$

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Theorem (Pyber,1993)

There are at most $n^{(2/27+o(1))\mu(n)^2} \leq 2^{\log_2^3 n}$ groups of order n.

L. Pyber, Enumerating finite groups of given order, *Ann. of Math.* 137 (1993), 203–220.

$|\{\text{a Cayley digraph of order } n\}| \leq 2^{n+\log_2^3 n} = 2^{n+o(n)}$.

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Lemma

For a transitive subgroup G of $\mathrm{Sym}(\Omega)$, there are at most $2^{3|\Omega|/4}$ digraphs Γ with vertex set Ω such that $G \leq \mathrm{Aut}(\Gamma)$.

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Since each automorphism group of a vertex-transitive digraph is 2-closed or contains a minimally transitive group,

$$\begin{aligned} &|\{\text{a vertex-transitive digraph of order } n\}| \\ \leq &|\{\text{a 2-closed transitive group of order } n\}| \cdot 2^{3n/4} \\ \text{or } \leq &|\{\text{a minimally transitive group of order } n\}| \cdot 2^{3n/4}. \end{aligned}$$

Polycirculant Conjecture

Polycirculant Conjecture: each vertex-transitive graph has a fixed-point-free element of prime order.

$\Rightarrow$ each vertex-transitive graph is a m-Cayley graph of a cyclic group.

Thank you for listening!