Rotary embeddings of Praeger-Xu graphs

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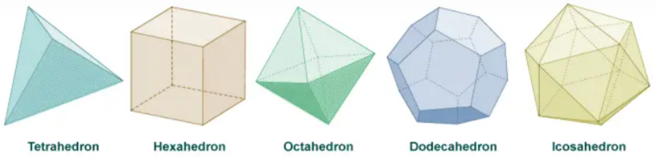

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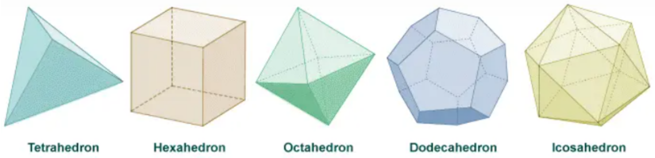

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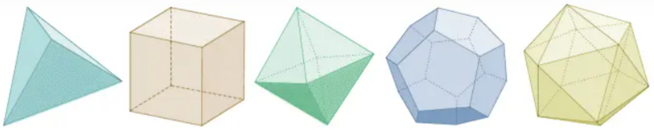


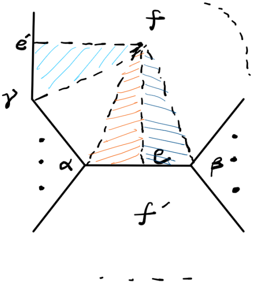
The face set:

The open disks $F := \mathcal{S} \setminus \Gamma$.

The flag set:

$\Phi := \{(\alpha, e, f) : \alpha \in V, e \in E, f \in F$ are mutually incident$\}$.

Remark: $|\Phi| = 4|E|$.


f
e'
γ
α
e
β
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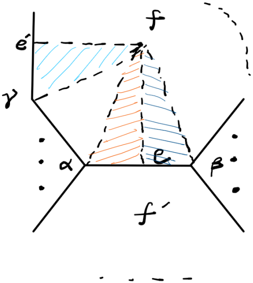

Automorphisms of map

$\mathrm{Aut}\mathcal{M} := \{\sigma \in \mathrm{Sym}(\Phi) : \sigma$ preserves incidences between flags$\}$.

1. $\mathrm{Aut}(\mathcal{M})_\phi = 1$, $\forall \phi \in \Phi \Rightarrow |\mathrm{Aut}\mathcal{M}|$ divides $|\Phi| = 4|E|$.
2. $\mathrm{Aut}(\mathcal{M})_\omega$ is cyclic or dihedral, $\omega \in V \cup E \cup F$. $\Rightarrow G_e \leqslant \mathrm{D}_4$.

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G-vertex (face)-rotary:

G_α (G_f) induces a transitive cyclic subgroup on edges that incident with the vertex α (the face f).

G-rotary map:

A map which is both G-vertex-rotary and G-face-rotary.

G: a finite group.

[0][15] C. H. Li, C. E. Praeger, and S. J. Song. Locally finite vertex-rotary maps and coset graphs with finite valency and finite edge multiplicity. *Journal of Combinatorial Theory. Series B*, 2024.

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RotaMap(G, ρ, τ)

A *rotary pair* $(\rho, \tau) \in G \times G$ of a group G satisfies $G = \langle \rho, \tau \rangle$ and $|\tau| = 2$. Define an incidence configuration RotaMap(G, ρ, τ) by

$$\text{vertex set } [G : \langle \rho \rangle], \text{ edge set } [G : \langle \tau \rangle] \text{ and face set } [G : \langle \rho\tau \rangle],$$

where two objects are incident if and only if their set intersection is non-empty.

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[15, Proposition 5.1]

A G-rotary map is isomorphic to RotaMap(G, ρ, τ) for some rotary pair (ρ, τ) for G.

[15, Proposition 4.1]

Two maps RotaMap(G, ρ, τ) and RotaMap(H, ρ', τ') are isomorphic if there is a group isomorphism $f : G \to H$ such that $f(\rho) = \rho'$ and $f(\tau) = \tau'$.

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- $H = \langle (\rho_1, \ldots, \rho_n), (\tau_1, \ldots, \tau_n) \rangle \leqslant \prod_{i=1}^{n} G_i$.

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Direct products

$\prod_{i=1}^{n} \mathcal{M}_i := \mathrm{RotaMap}\left(H, (\rho_1, \ldots, \rho_n), (\tau_1, \ldots, \tau_n)\right)$.

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Praeger-Xu graph

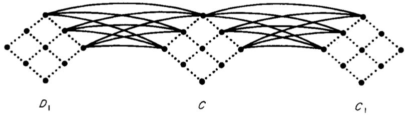
Let p, r, s be positive integers such that $p \geqslant 2$ and $r \geqslant 3$. Define a simple graph $\mathrm{C}(p, r, s) = (V, E)$ as follows:

1. the vertex set V is $\mathbb{Z}_r \times \mathbb{Z}_p^s$;
2. the edge set E is defined to be the set of all pairs of the form

$$\{(i, x_0, x_1, \ldots, x_{s-1}), (i+1, x_1, \ldots, x_{s-1}, x_s)\}$$

for every $i \in \mathbb{Z}_r$ and $x_0, x_1, \ldots, x_{s-1}, x_s \in \mathbb{Z}_p$.

There are $p^{s+1}r$ edges.



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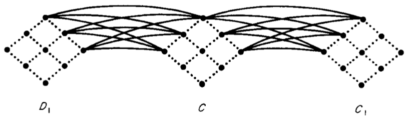

Augmented PX-graphs

PX-graphs, multicycles $\mathrm{C}_r^{(p)}$ (denoted by $\mathrm{C}^*(p, r, 0, 1)$), and cycles C_{pr} (denoted by $\mathrm{C}^*(p, r, 0, -1)$).

[19, Theorem 2.10]

The graph $\mathrm{C}(p, r, s)$ is symmetric if and only if $r \geqslant s + 1$, and is vertex transitive if and only if $r \geqslant s$.

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[19, Theorem 2.13]

The full automorphism group of $\mathrm{C}(p, r, s)$ is equal to S_p *wr* D_{2r} where $(r, s) \neq (4, 1)$, $r \geqslant \max\{s + 1, 3\}$ and p is odd.

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[19, Theorem 1]

Let Γ be a connected, simple, G-arc-transitive graph of valency $2p$. If G contains an abelian normal p-subgroup which is not semiregular on the vertices of Γ, then $\Gamma = \mathrm{C}(p, r, s)$ for some $r \geqslant \max\{3, s + 1\}$ and $s \geqslant 1$.

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Case 1. $(r, s) \neq (4, 1)$

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Suppose that G is an arc-regular subgroup of $\mathrm{Aut}(\mathrm{C}(p, r, s))$. Then

$$G \cong \mathbb{Z}_p^{s+1}{:}\mathrm{D}_{2r}.$$

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It follows from $(\mathrm{S}_p^r)_p = \mathbb{Z}_p^r$ that

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Let $p = 3$. By Brodkey's theorem, there are two Sylow 3-subgroups P and Q of L such that $P \cap Q = \mathrm{O}_3(L) \cong \mathbb{Z}_3$. The quotient group $L/\mathrm{O}_3(L) \cong \mathrm{A}_4$ since its Sylow 3-subgroups are not normal.

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Theorem

Let $s \geqslant 0$ be an integer, let $\delta \in \{1, -1\}$ and let r be an integer such that $r \geqslant \max\{3, s+1\}$. Let G be an arc-regular group of automorphisms of $\mathrm{C}^*(p, r, s, \delta)$. Then $G \cong \mathbb{Z}_p^{s+1}{:}\,\mathrm{D}_{2r}$.

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Let $\mathcal{M}$ be a G-rotary map with underlying graph isomorphic to $\mathrm{C}^*(p, r, s, \delta)$. Then G is isomorphic to $\mathbb{Z}_p^{s+1}{:}\,\mathrm{D}_{2r}$.

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$$|G_{\alpha\beta}| = \begin{cases} 2 \Rightarrow r = 2; \\ p \Rightarrow \text{the underlying graph is the multi-cycle } \mathrm{C}^*(p, r, 0, 1) \\ 2p \Rightarrow r = 2; \end{cases}$$

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Proposition

Let $x, y \in D$ be involutions such that $D = \langle x, y \rangle$, and let $v \neq v' \in \mathsf{C}_V(x) \backslash \{1\}$. Then $\mathrm{RotaMap}(G, vx, y) \cong \mathrm{RotaMap}(G, v'x, y)$ are rotary PX maps

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$$\begin{aligned} f(w_1g_1w_2g_2) &= f(w_1w_2^{g_1^{-1}}g_1g_2) \\ &= \sigma(w_1)\sigma(w_2^{g_1^{-1}})g_1g_2 \\ &= \sigma(w_1)(\sigma(w_2))^{g_1^{-1}}g_1g_2 \\ &= \sigma(w_1)g_1\sigma(w_2)g_2 \\ &= f(w_1g_1)f(w_2g_2). \end{aligned}$$

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ρ_x

For any $x \in D \leqslant G = V{:_\psi}D$, let ρ_x denote an element in the set

$$\{vx \mid 1 \neq v \in \mathrm{C}_V(x)\}.$$

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Let $x_1, y_1, x_2, y_2 \in D$ be involutions where $D = \langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle$. Let $\sigma \in \mathrm{Aut}(D)$ be given by $\sigma(x_1) = x_2$ and $\sigma(y_1) = y_2$. Then

1. $\mathsf{RotaMap}(G, \rho_{x_1}, y_1) \cong \mathsf{RotaMap}(G, \rho_{x_2}, y_2)$ if and only if $\psi \cong \psi \circ \sigma$.
2. Define $\mathcal{RM}_G := \{\mathsf{RotaMap}(G, \rho_x, y) \mid \langle x, y \rangle = D\}$. The mapping

$$\begin{aligned} \psi^{\mathrm{Aut}(\mathrm{D}_{2r})} &\to \mathcal{RM}_G, \\ \psi \circ \eta &\mapsto \mathsf{RotaMap}(G, \rho_{\eta(x_1)}, \eta(y_1)) \end{aligned}$$

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$\Rightarrow$: Let $f \in \mathrm{Aut}(G)$ be the isomorphism. Since x_1 and x_2 are involutions, there is

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So $\sigma := f|_D \in \mathrm{Aut}(D)$. Straightforward checking shows that $f|_V$ serves as the isomorphism between $\psi \circ \sigma$ and ψ.

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Note that $\mathrm{Aut}(D)$ acts transitively on the set $\mathcal{RM}_G$. There is $|\mathcal{RM}_G| = |\mathrm{Aut}(D)|/|\mathrm{Aut}(D)_\psi| = |\psi^{\mathrm{Aut}(D)}|$, which completes the proof.

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r	$\mathrm{Irr}(D_{2r})$	G	Γ	the count
r is odd	$\gamma_{1,1}$	$\mathbb{Z}_p \times \mathrm{D}_{2r}$	$\mathrm{C}^*(p, r, 0)$	1
	$\gamma_{-1,-1}$	D_{2pr}	$\mathrm{C}^*(p, r, 0, -1)$	1
	ψ ($d \geqslant 2$)	$\mathbb{Z}_p^d \rtimes_\psi \mathrm{D}_{2r}$	$\mathrm{C}(p, r, d-1)$	$\lvert\psi^{\mathrm{Aut}(\mathrm{D}_{2r})}\rvert$
r is even	$\gamma_{1,1}$	$\mathbb{Z}_p \times \mathrm{D}_{2r}$	$\mathrm{C}^*(p, r, 0)$	1
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Table 1: Irreducible G-rotary PX maps

[2, Theorem 5.7]

The number of irreducible rotary augmented PX map of length r is

1. $|\mathrm{Irr}(D_{2r})|$ if r is odd;
2. $|\mathrm{Irr}(D_{2r})| - 1$ if r is even.

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So there are $p^{d/2}\varphi(r)$ choices of τ. $\square$

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Since $\mathrm{End}_D(V) \cong \mathbb{F}_p^{d/2}$, we have that the number of G-rotary augmented PX maps equals $\varphi(r)/|\mathrm{N}_A(D)| = |\mathrm{Aut}(D)|/|\mathrm{Aut}(D)_{\psi} = |\psi^{\mathrm{Aut}(D)}|$.

Sketch of the proof

For each $\delta \in \mathrm{Aut}(D)_\psi$, there exists $\sigma \in \mathrm{Aut}(G)$ such that $\sigma\mid_D = \delta$.Fix such an automorphism δ of D, then there exists an $\mathbb{F}_p D$-module isomorphism δ_V such that $\delta_V(\psi(b)(v)) = \psi \circ \delta(b)\delta_V(v)$ for each $b \in D$ and $v \in V$. Define

$$\sigma : G \to G,\ vb \mapsto \delta_V(v)\delta(b),$$

which is an automorphism of G.Thus $\mathrm{im}(\kappa) = \mathrm{Aut}(D)_\psi$.

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[2, Lemma 5.2]

For $i \in \{1, 2\}$, let $G_i = \mathbb{Z}_p^{d_i}{:}_{\psi_i} D_i$, where $p \nmid |D_1||D_2|$. Then the following are equivalent:

1. $G_1 \cong G_2$;
2. there exists an isomorphism $\sigma : D_1 \to D_2$ such that $\psi_2 \circ \sigma \cong \psi_1$.

Moreover, if (ii) holds, there exists an isomorphism $f : G_1 \to G_2$ such that $f(D_1) = D_2$ and $f|_{D_1} = \sigma$.

[2, Theorem 6.2] (the existence of decompositions)

Let $\mathcal{M}$ be a rotary augmented PX map of length r. Then $\mathcal{M}$ is isomorphic to a direct product of irreducible rotary augmented PX maps of length r. Conversely, every direct product of irreducible rotary augmented PX maps of length r is a rotary augmented PX map whose underlying graph is of length r.

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[2, Lemma 6.4]

Let $V_1 = \mathbb{Z}_p^{d_1}$ and let $V_2 = \mathbb{Z}_p^{d_2}$. Let $G = (V_1 \times V_2){:}_{(\psi_1,\psi_2)} \mathrm{D}_{2r}$ where $\psi_i : \mathrm{D}_{2r} \to \mathrm{GL}(V_i)$ is irreducible for every $i \in \{1, 2\}$. If G has a rotary pair (ρ, τ) such that $|\rho| = 2p$ and $|\tau| = 2$, then $\psi_1 \ncong \psi_2$.

Sketch of the proof

Suppose the contradiction. There are $v \in V_1 \times V_2$, $\tau_1 \in \mathrm{D}_r$ such that $\rho = v\tau_1$.

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Theorem

Let $\mathcal{M}_1, \ldots, \mathcal{M}_m, \mathcal{N}_1, \ldots, \mathcal{N}_n$ be irreducible rotary augmented PX maps whose underlying graph is of length r. Let $\mathcal{M}_i \ncong \mathcal{M}_j$ for $i \neq j$ and let $\mathcal{N}_i \ncong \mathcal{N}_j$ for $i \neq j$. Then $\mathcal{M}_1 \times \cdots \times \mathcal{M}_m \cong \mathcal{N}_1 \times \cdots \times \mathcal{N}_n$ if and only if $m = n$ and there is a permutation σ of $\{1, \ldots, n\}$ such that $\mathcal{M}_i \cong \mathcal{N}_{\sigma(i)}$ for all i.

1. Map

2. Rotary Praeger-Xu maps (PX maps)
   - Praeger-Xu graphs
   - The characterization of arc-regular automorphism groups of Praeger-Xu graphs

3. Irreducible rotary PX maps
   - Construction of a class of irreducible rotary PX maps of length r
   - The correspondence between irreducible rotary PX maps of length r and irreducible representations of D_{2r}
   - The count of $V{:}_{\psi}\mathrm{D}_{2r}$-rotary PX maps

4. The decomposition of rotary PX maps

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