Supersolvable Fusion Systems

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Outline

Background

Definition 1

Given a finite group G and a p-subgroup $S \in Sylow_p(G)$, $\mathcal{F}_S(G)$ is used to represent the fusion category of G on S: the objects of $\mathcal{F}_S(G)$ are all subgroups of S and morphisms in $\mathcal{F}_S(G)$ are the group homomorphisms between subgroups of S induced by conjugation in G.

$Hom_G(H, K) = \{\varphi \in Hom(H, K) | \varphi = c_g$ for some $g \in G$ such that $H^g \leq K\}$.
Then we have $Mor_{\mathcal{F}_S(G)}(P, Q) = Hom_G(P, Q)$.

Example 2

$G = S_4$, $S = C_3$. Then $Ob(\mathcal{F}_S(G)) = \{C_3, 1\}$, $Mor_{\mathcal{F}_S(G)}(C_3, C_3) = \{1_{C_3}, -1_{C_3}\}$, $Mor_{\mathcal{F}_S(G)}(1, C_3) = \{1\}$, $Mor_{\mathcal{F}_S(G)}(1, 1) = \{1\}$.

And we can remove G and define the abstract fusion systems.

Definition 3

A fusion system over a p-group S is a category $\mathcal{F}$, where $Ob(\mathcal{F})$ is the set of all subgroups of S, and which satisfies the following two properties for all $P, Q \leq S$:

- $Hom_S(P, Q) \subseteq Mor_{\mathcal{F}}(P, Q) \subseteq Inj(P, Q)$;
- each $\varphi \in Mor_{\mathcal{F}}(P, Q)$ is the composite of an $\mathcal{F}$-isomorphism followed by an inclusion.

A fusion system $\mathcal{F}$ on S is called exotic if there is no group G such that $\mathcal{F} = \mathcal{F}_S(G)$.

[1] Aschbacher, Michael G., Kessar, Radha, Oliver, Bob. Fusion systems in algebra and topology. Cambridge University Press, 2011.

[2] David A. Craven. The theory of fusion systems, volume 131 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2011. An algebraic approach.

In 2006, Puig introduced saturated fusion systems [3], and defined nilpotent fusion systems that coincide with p-nilpotency in finite groups.

Definition 4

A fusion system $\mathcal{F}$ over a p-group S is saturated if each subgroup of S is $\mathcal{F}$-conjugate to a subgroup which is fully automized and receptive.

Theorem 5

For a finite group G and its Sylow-p subgroup S, $\mathcal{F}_S(G)$ is saturated.

Theorem 6 (Alperin's fusion theorem)

Fix a p-group S and a saturated fusion system $\mathcal{F}$ over S. Then

$$\mathcal{F} = \langle Aut_{\mathcal{F}}(P) | P = S \text{ or } P \text{ is } \mathcal{F}\text{-essential} \rangle_S.$$

[3] Lluis Puig. Frobenius categories. J. Algebra, 303(2006), 309-357.

Definition 7

A saturated fusion system $\mathcal{F}$ on S is nilpotent if $\mathcal{F} = \mathcal{F}_S(S)$.

Theorem 8

For a finite group G and its Sylow-p subgroup S, $\mathcal{F}_S(G)$ is nilpotent if and only if G is a p-nilpotent group.

Since then, researchers in group theory began to incorporate the previous concepts and conclusions from the groups into fusion systems.

[3] Lluis Puig. Frobenius categories. J. Algebra, 303(2006), 309-357.

Supersolvable fusion systems

In 2018 and 2022, Su[4], Zhang and Shen[5] respectively defined the supersolvable fusion system.

Definition 9

Let S be a p-group and $\mathcal{F}$ be a saturated fusion system on S. If there exists a subgroup sequence of S, $1 = S_0 \leq \cdots \leq S_m = S$, such that for each $i \in \{0, \cdots, m-1\}$, S_i is strongly closed in S with respect to $\mathcal{F}$, and S_{i+1}/S_i is cyclic, then $\mathcal{F}$ is called supersolvable.

[4] Su, Ning. On supersolvable saturated fusion systems, Monatsh. Math. **187**, no. 1, 171–179, 2018.

Definition 10

A fusion system $\mathcal{F}$ is called supersolvable if there exists a normal series $\mathcal{E}_0 = 1 \trianglelefteq \mathcal{E}_1 \trianglelefteq \mathcal{E}_2 \trianglelefteq \cdots \trianglelefteq \mathcal{E}_n = \mathcal{F}$ within $\mathcal{F}$, such that $\mathcal{E}_j/\mathcal{E}_i \cong \mathcal{F}_{Z_p}(Z_p)$ or is trivial $(1 \leq i < j \leq n)$, where Z_p is a p-order cyclic group.

This definition is equivalent to Su's definition.

When G is a p-supersolvable group, $\mathcal{F}_S(G)$ is a supersolvable fusion system. However, the converse is not true.

Example 11

Let $G = PSL(2, 17)$ and S be a 3-Sylow subgroup of G. Then $|S| = 9$ and $\mathcal{F}_S(G)$ is supersolvable, but G is not p-supersolvable.

[5] Shen, Zhencai, Zhang, Jiping. *p*-supersolvable fusion systems(in Chinese). Scientia Sinica Mathematica, 1113–1120, 2022.

In 2024, Fawaz and Julian obtained some results regarding the supersolvable fusion system.

Theorem 12

Let G be a finite group, p be a prime and S be a Sylow p-subgroup of G. Suppose that there is a subgroup D of S with $1 < D < S$ such that any subgroup of S with order $|D|$ or $p|D|$ is abelian and weakly pronormal in G. Then $\mathcal{F}_S(G)$ is supersolvable.

[6] Aseeri, Fawaz, Kaspczyk, Julian, Criteria for supersolvability of saturated fusion systems, J. Algebra, 647: 910–930, 2024.

Our target is to explore how do normal subgroups or complement subgroups affect the supersolvability of $\mathcal{F}_S(G)$.

Theorem 13

Suppose G is finite and $S \leq G$ is a Sylow p-subgroup where p is an odd prime. If all subgroups of S with order p are normal in G or have normal complement in G. Then $\mathcal{F}_S(G)$ is supersolvable.

Theorem 14

Let G be a finite group and $S \leq G$ be a Sylow p-subgroup where p is an odd prime. If all maximal subgroups of S are normal in G or have normal complement in G. Then $\mathcal{F}_S(G)$ is supersolvable.

Theorem 15

Suppose G is a group and $S \in Syl_p(G)$ where p is an odd prime dividing $|G|$. Assume S has a subgroup U satisfying $1 < |U| < |S|$ and any subgroup of S with order $|U|$ or $p|U|$ is normal in G or has normal complement in G. Then $\mathcal{F}_S(G)$ is supersolvable.

Question 1

Let S be a p-group and $\mathcal{F}$ be a saturated fusion system on S. Suppose that there is a subgroup D of S with $1 < D < S$ such that any subgroup of S with order $|D|$ is weakly $\mathcal{F}$-closed. If S is not cyclic, suppose moreover that S has more than one abelian subgroup with order $|D|$. Is it true in general that $\mathcal{F}$ is supersolvable?

More research questions about fusion system

Remark 1

[7] Let p be an odd prime and let S be a rank two p-group. Given a saturated fusion system $(S, \mathcal{F})$, one of the following holds:

- *$\mathcal{F}$ has no proper $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups, and it is the fusion system of the group $S : Out_{\mathcal{F}}(S)$*
- ...

Remark 2

[8] What kind of fusion systems will be exotic?

[7] A. Díaz, A. Ruiz, A. Viruel. All p-local finite groups of rank two for odd prime p. Trans. Amer. Math. Soc. 1725–1764, 2007.

[8] A. Ruiz, and A. Viruel. The classification of p-local finite groups over the extraspecial group of order p^3 and exponent p. Math. Z. 248, no. 1, 45–65, 2004.

Thank You!

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[3] Lluis Puig. Frobenius categories. J. Algebra, 303(2006), 309-357.

[4] Su, Ning. On supersolvable saturated fusion systems, Monatsh. Math. **187**, no. 1, 171–179, 2018.

[5] Shen Z, Zhang J. p-supersolvable fusion systems(in Chinese). Scientia Sinica Mathematica, 1113–1120, 2022.

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[6] F. Aseeri, J. Kaspczyk. Criteria for supersolvability of saturated fusion systems. J. Algebra, 647: 910–930, 2024.

[7] Díaz, Antonio, Ruiz, Albert and Viruel, Antonio. All p-local finite groups of rank two for odd prime p. Trans. Amer. Math. Soc. 1725–1764, 2007.

[8] Ruiz, Albert, and Viruel, Antonio. The classification of p-local finite groups over the extraspecial group of order p^3 and exponent p. Math. Z. 248, no. 1, 45–65, 2004.

[9] Zhang S, Shen Z. Theorems of Szép, Zappa and Gaschütz for fusion systems, submitted.