Which Maximal Subgroups are Perfect Codes

Zhishuo Zhang

This is a joint work with
Shouhong Qiao, Ning Su, Binzhou Xia, and Sanming Zhou.

University of Melbourne

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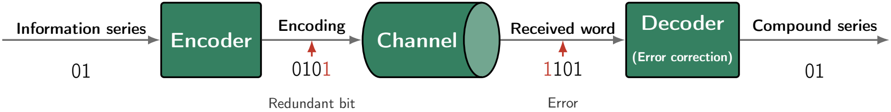
Error-Correcting Codes

The following diagram shows a process of information transmission.



- A code of length n over a set F is a subset $C \subseteq F^n$.
- The Hamming-distance: $d(x, y) := |\{i \mid x_i \neq y_i\}|$.
- Closed ball of radius t: $B_t(x) = \{y \mid d(x, y) \leq t\}$.
- A code C is a t-error-correcting code if
$$\forall x, y \in C, \ x \neq y \Rightarrow B_t(x) \cap B_t(y) = \varnothing.$$
- A t-error-correcting code is perfect if $\bigcup_{x \in C} B_t(x) = F^n$.

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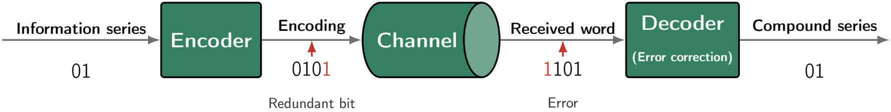

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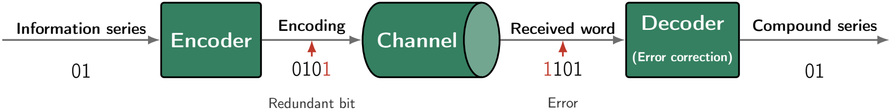

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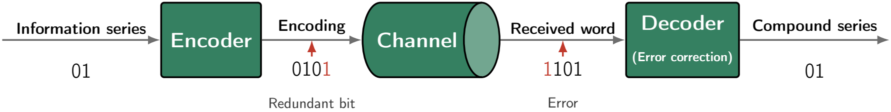

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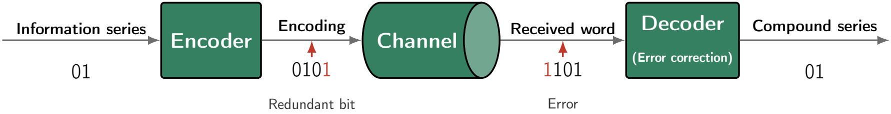

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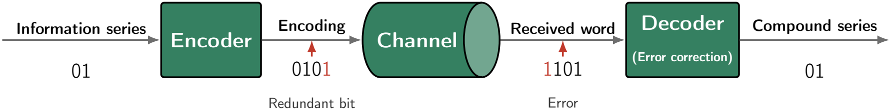

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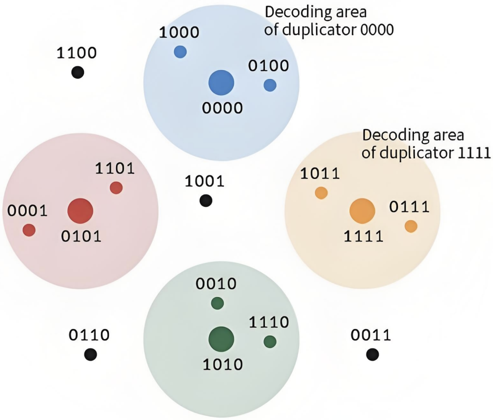

The receive space

Perfect Codes on Distance-Transitive Graphs

Biggs[1] (1973) generalized the setting of perfect t-codes to graphs:

- Let Γ be a graph with vertex set F^n;
- For distinct $x, y \in V(\Gamma)$, $x \sim y$ iff $d(x, y) = 1$.

The proper setting for perfect code question is distance-transitive graphs.

Lloyd's Thereom[2]

If a perfect t-error-correcting code exists in F^n, where $|F| = q$, then

$$\sum_{k=0}^{t}\sum_{j=0}^{k}(-1)^j(q-1)^{k-j}\binom{x}{j}\binom{n-x}{k-j}$$

has t distinct integral zeros among $1, \ldots, n$.

[1] N. L. Biggs, Perfect codes in graphs, *J. Combinatorial Theory, Ser. B*, 15 (1973), 289–296.
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Perfect 1-Codes on Graphs

Definition

A perfect code in a graph $\Gamma = (V, E)$ is a subset C of V such that no two vertices in C are adjacent and every vertex in $V \backslash C$ is adjacent to exactly one vertex in C.

Closed neighborhoods of all vertices in C forms a partition of V.

Hamming codes are perfect codes on Hamming graphs $H(n, q)$:

- $V = \mathbb{F}_q^n$,
- $(x_1 \ldots, x_n) \sim (y_1, \ldots, y_n)$ iff they differ in exactly one coordinate.

$H(n, q)$ is a Cayley graph.

There has been considerable research on perfect codes in Cayley graphs.

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Codes Equipped with Algebraic Structure

Let F be a set and $C \subseteq F^n$ be a code.

- Linear code: $F = \mathbb{F}_q$ and C is a linear subspace of $\mathbb{F}_q^n$.
- Group code: F^n is an additive group and $C \leq F^n$.

Definition

If a perfect code H of a Cayley graph $\mathrm{Cay}(G, S)$ is a subgroup of G, then H is called a subgroup perfect code.

The additional subgroup structure

- enriches the theoretic study
- offers advantages in efficient representation and computation

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Subgroup Perfect Codes of a Group

Problem

Classify (H, S) such that H is a subgroup perfect code of $\mathrm{Cay}(G, S)$.

A natural starting point is to determine which subgroups $H \leq G$ admit such a pair.

Definition (Huang–Xia–Zhou[1], 2018)

H is called a subgroup perfect code of G if H is a subgroup perfect code of some Cayley graph on G.

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Characterization of Subgroup Perfect Codes

Theorem (Chen–Wang–Xia[1], 2020)

Let $H \leq G$. Then the following are equivalent:

- H pc G;
- there exists an inverse-closed left transversal of H in G;
- for each $a \in G$ such that $a^2 \in H$ and $|H|/|H \cap H^a|$ is odd, there exists $b \in aH$ such that $b^2 = e$;
- for each $a \in G$ such that $HaH = Ha^{-1}H$ and $|H|/|H \cap H^a|$ is odd, there exists $b \in aH$ such that $b^2 = e$.

[1] J. Chen, Y. Wang and B. Xia, Characterization of subgroup perfect codes in Cayley graphs. *Discrete Math.* 343 (2020), no. 5, 111813, 4 pp.

Characterizing Subgroup Perfect Codes by 2-Subgroups

Theorem (Zhang[1], 2023)

Let $H \leq G$, and let $Q \in \mathrm{Syl}_2(H)$. Then the following are equivalent:

- H is a perfect code of G;
- Q is a perfect code of G;
- Q is a perfect code of any Sylow 2-subgroup of $\mathrm{N}_G(Q)$;

[1] J. Zhang, Characterizing subgroup perfect codes by 2-subgroups. *Des. Codes Cryptogr.* 91 (2023), no. 9, 2811–2819.

2-Local Prespective

Theorem (Qiao–Su–Xia–Z.–Zhou, 2025+)

Let H be a 2-subgroup of G. Then H pc G iff, for each $a \in \mathrm{N}_G(H) \setminus H$ with $a^2 \in H$, the subgroup H has a complement in $H\langle a \rangle \cong H.C_2$.

Let $H \cong C_{2^n} \rtimes C_2$, and G be an arbitrary group containing H. We characterized whether H pc G based on the above theorem.

Similar method can be used to study other groups.

Proposition (Qiao–Su–Xia–Z.–Zhou, 2025+)

Let $G = \mathrm{SL}_2(q)$ with prime power q. For even q, every subgroup of G is a perfect code. For odd q, H pc G iff either $|H|_2 = 1$ or $|H|_2 = |G|_2$.

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Maximal Subgroup Perfect Codes

Lemma (Zhang–Zhou[1], 2021)

If $H \leq M \leq G$, then H pc $G \Longrightarrow H$ pc M.

To determine whether H pc G, we begin by examining whether H pc M, where M is the smallest overgroup properly containing H.

In other words, we focus on

Problem

For $H \underset{\max}{<} G$, whether H pc G?

A good perfect code should have large cardinality in order to make efficient use of the channel.

[1] J. Zhang and S. Zhou, Corrigendum to "On subgroup perfect codes in Cayley graphs" [*European J. Combin.*, 91 (2021) 103228], *European J. Combin.* 101 (2022), Paper No. 103461, 5 pp.

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Relevant Results in the Literature

G is called code-perfect if $\forall H < G$, H pc G.

Theorem (Ma–Walls–Wang–Zhou[1], 2020)

A group is code-perfect iff it has no elements of order 4.

Corollary (Ma–Walls–Wang–Zhou[1], 2020)

A simple group is code-perfect iff it is isomorphic to one of:

- C_p, where p is a prime;
- $\mathrm{PSL}_2(2^e)$, $e \geq 2$;
- $\mathrm{PSL}_2(q)$, $q \equiv \pm 3 \pmod 8$, $q > 5$;
- a Ree group ${}^2G_2(3^{2n+1})$, $n > 1$;
- the Janko group J_1.

[1] X. Ma, G. L. Walls, K. Wang and S. Zhou,Subgroup Perfect Codes in Cayley Graphs. *SIAM J. Discrete Math.* 34 (2020), no. 3, 1909–1921.

Relevant Results in the Literature

Theorem (Chen–Li–Zhang[1], 2025)

If G is one of the following groups:

- $\mathrm{PSL}_2(q)$, where q is a prime power;
- $\mathrm{Suz}(q)$, where $q = 2^{2m+1}$;
- $\mathrm{PSU}_3(q)$, where q is a prime power,

and $H \underset{\max}{<} G$, then the sufficient and necessary condition for H pc G is classified.

[1] Z. G. Chen, J. J. Li and J. Y. Zhang, Subgroup perfect codes in Lie type simple groups of rank one, *Des. Codes Cryptogr.* (2025), 1–18.

Relevant Results in the Literature

Theorem (Bu–Li–Zhang[1], 2025)

The sufficient and necessary condition of H pc G is classified for the following (G, H):

- $G = C_{2^n}.C_2$ and $H < G$;
- $G = \mathrm{SL}_3(q)$, where q is a prime power with $q \equiv -1 \pmod 4$, and $H \underset{\max}{<} G$.

[1] X. C. Bu, J. J. Li and J. Y. Zhang, Subgroup perfect codes of 2-groups with cyclic maximal subgroups, *Bull. Malays. Math. Sci. Soc.* 48 (2025), no. 3, Paper No. 78, 9 pp.

Quotient Reduction

Lemma (Zhang–Zhou[1], 2021)

Let $N \trianglelefteq G$ and $N \leq H \leq G$. Then H pc $G \Longrightarrow H/N$ pc G/N.

- Given a group G with $H \underset{\max}{<} G$, and let $N = \mathrm{Core}_G(H)$;
- Then $H/N \underset{\max}{<} G/N$, and so G/N is primitive.

Reduced Problem

Let H be the stabilizer of a primitive group G. Whether H pc G?

Lifting Problem

H/N pc $G/N \Longrightarrow H$ pc G?

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[1] J. Zhang and S. Zhou, Corrigendum to "On subgroup perfect codes in Cayley graphs" [*European J. Combin.*, 91 (2021) 103228], *European J. Combin.* 101 (2022), Paper No. 103461, 5 pp.

Quotient Reduction

Lemma (Zhang–Zhou[1], 2021)

Let $N \trianglelefteq G$ and $N \leq H \leq G$. Then H pc $G \Longrightarrow H/N$ pc G/N.

- Given a group G with $H \underset{\max}{<} G$, and let $N = \mathrm{Core}_G(H)$;
- Then $H/N \underset{\max}{<} G/N$, and so G/N is primitive.

Reduced Problem

Let H be the stabilizer of a primitive group G. Whether H pc G?

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Lemma (Qiao–Su–Xia–Z.–Zhou, 2025+)

If G is a split extension of N by G/N, then H/N pc $G/N \implies H$ pc G.

Question

Is there any better characterization of whether

$$H/N \text{ pc } G/N \implies H \text{ pc } G\ ?$$

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Perfect codes in Primitive Groups

Theorem (Qiao–Su–Xia–Z.–Zhou, 2025+)

Let G be a primitive group of type HA, HS, HC, TW, SD or CD, and let H be a point stabilizer. Then H pc G.

Diamond Lemma (Qiao–Su–Xia–Z.–Zhou, 2025+)

Let H and K be subgroups of a group G such that HK is a group. Then $(H \cap K)$ pc $K \implies H$ pc HK.

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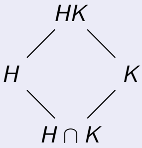

Type AS

Theorem (Qiao–Su–Xia–Z.–Zhou, 2025+)

Let $T = \mathrm{PSL}_2(q)$ with prime power $q \geq 4$, let G be a primitive almost simple group with socle T, and let H be the point stabilizer of G. Then H not pc G iff one of the following holds:

- $q > 7$, $q \equiv -1 \pmod 8$, $|G/T|$ is odd, and $H \cap T \cong D_{q-1}$;
- $q > 9$, $q \equiv 1 \pmod 8$, $|G/T|$ is odd, and $H \cap T \cong D_{q+1}$;
- $q \equiv 1 \pmod 8$, $|G/T| \equiv 2 \pmod 4$, $G \not\leq \mathrm{P\Sigma L}_2(q)$, $G \not\geq \mathrm{PGL}_2(q)$, and $H \cap T \cong D_{q+1}$.

Similar method can be used to consider almost simple groups with other socle.

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Type AS

Certain almost simple groups, such as $\mathrm{PGL}_2(q)$, have the property that every maximal subgroup is a perfect code.

Question (Qiao–Su–Xia–Z.–Zhou, 2025+)

Which almost simple groups have the property that every maximal subgroup is a perfect code?

Question (Xia–Zhang–Z.[1], 2025+)

Whether every maximal subgroup of S_n is a perfect code?

The only known results on this question are

- intransitive maximal subgroups of S_n;
- $\mathrm{AGL}_1(p)$ as maximal subgroups of S_p for odd primes p;
- $\mathrm{AGL}_2(p)$ as maximal subgroups of S_{p^2} for $p \equiv 3 \pmod 4$.

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Type PA

Proposition (Qiao–Su–Xia–Z.–Zhou, 2025+)

For a permutation group K such that $H \wr K \leq G \wr K$,

$$H \text{ pc } G \implies H \wr K \text{ pc } G \wr K.$$

The converse does not hold, which means the problem of type PA can not be reduced to the problem of type AS.

Theorem (Qiao–Su–Xia–Z.–Zhou, 2025+)

Let $H \leq G$. If $|H|_2 \leq 2$, then $H \wr S_2$ is a perfect code of $G \wr S_2$.

Corollary (Qiao–Su–Xia–Z.–Zhou, 2025+)

Let $T = \mathrm{PSL}_2(q)$ with prime power q, and let $H \underset{\max}{<} T$. Then

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Further Research

Question

Is there any better characterization of whether

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Question

What more can we say about types AS and PA?

Question

Which almost simple groups have the property that every maximal subgroup is a perfect code?

Question

Whether every maximal subgroup of S_n is a perfect code?

Thank You.