On fixer of finite Ree group

with socle ${}^2G_2(q)$

Let $G \leq Sym(\Omega)$ be finite and transitive

$K \leq G$ is called a fixer if each elements in K fixes some point in Ω

$\Leftrightarrow$ $K \leq G$ is a fixer if $\forall k \in K$, k is conjugate to some element in G_ω.

For the remaining. Let $G_0 = {}^2G_2(q)$, where $q = 3^{2n+1}$
$G_0 \trianglelefteq G \trianglelefteq Aut(G_0)$

Theorem: Let G be primitive on Ω. if $K \leq G$ is a fixer with $|K| \geq |G_\omega|$ then

(G_ω, K) is one of the following.

i)

ii)

$[G_2(K)$ Lie $^2G_2(K)]$

Let $\mathbb{L}$ be a simple Lie algebra of type G_2 over K, with root system Φ.

$\mathbb{L} = \underline{\mathbb{H}} \oplus \mathbb{L}_{r_1} \oplus \cdots \oplus \mathbb{L}_{r_k}$ be a Cartan decomposition.
and $\{h_r, e_r, r \in \Phi\}$ be a Chevalley basis, Ref Chapter 4.1 Carter

The Chevalley group of type G_2 over K. $G_2(K)$ is the subgroup of the automorphism group of $\mathbb{L}$ generated by $\{\underline{x_r(t)}, r \in \Phi, t \in K\}$,

where $x_r(t) = \exp(t\,ad\,e_r) = 1 + t\,ad\,e_r + \frac{t^2}{2}(ad\,e_r)^2 + \cdots + \frac{(ad\,e_r)^n}{n!} t \cdots$

References

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[graph automorphism of $G_2(K)$]

Char $K = 3$

There is a permutation on Φ $r \longrightarrow \bar{r}$, obtained by reflecting the line bisecting a and b.

Then $\tau: x_r(t) \longrightarrow x_{\bar{r}}(t^{(r,r)})$ (where (r,r) is the inner-product $\mathrm{Tr}(\mathrm{ad}r \cdot \mathrm{ad}r)$)

τ extends to an automorphism of $G_2(K)$.

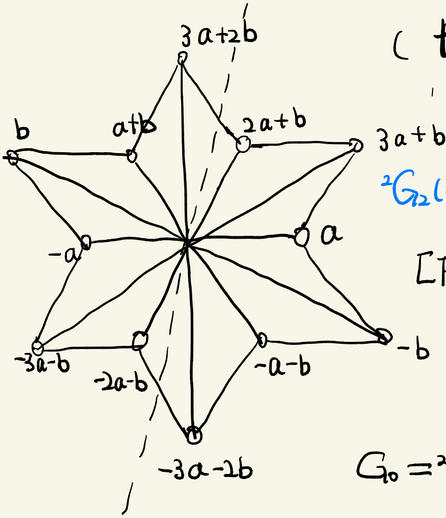
[pf]: τ preserves the Chevalley relations (ref. Prop 12.2.1) Carter.

[Field automorphism].

Let f be an automorphism of K

Then $f: x_r(t) \longrightarrow x_r(t^f)$ extends to an automorphism of $G_2(K)$, called the field automorphism.

[Lemma]: $\tau f = f\tau$. Let $\underline{\theta}$ be an field automorphism of $G_2(K)$ which maps $x_r(t)$ to $x_r(t^{3^n})$ ($t^\theta = t^{3^n}$).



${}^2G_2(K) = \{ g \in G_2(K) \mid \tau \cdot \theta(g) = g \}$

${}^2G_2(q)$ $K = \mathbb{F}_{3^{2n+1}} = \mathbb{F}_q$

[Fact]: For any field automorphism f of $G_2(K) = G_2(q)$,

$${}^2G_2(q)^f = {}^2G_2(q)$$

Ref: Low-dim Table.

$G_0 = {}^2G_2(q) \trianglelefteq G \trianglelefteq \mathrm{Aut}(G_0) = \boxed{G_0 : \langle \phi \rangle}$

where ϕ maps $x_r(t)$ to $x_r(t^3)$

[Borel subgroup of $G_0 = {}^2G_2(q)$].

Let U be the subgroup of $G_2(q)$ gererated by $\{x_r(t), r \in \Phi^+, t \in \mathbb{F}_q\}$, then U is called a maximal unipotent subgroup of $G_2(k)$

Then $Q = U \cap {}^2G_2(q)$ is a maximal unipotent subgroup of $G_0 = {}^2G_2(q)$ (which is also a Sylow 3-subgroup of G_0)

$\alpha(t) = x_a(t^\theta) x_b(t) x_{a+b}(t^{\theta+}) x_{2a+b}(t^{2\theta+1})$

$\beta(u) = x_{a+b}(u^\theta) x_{3a+b}(u)$ Carter P 236.

$\gamma(v) = x_{2a+b}(v^\theta) x_{3a+2b}(v)$

$x_S(t, u, v) = \alpha(t)\beta(u)\gamma(v)$, where $S = \{a, b, a+b, 2a+b, 3a+b, 3a+2b\}$ is an equivalence class

$Q = \{x_S(t, u, v) \mid t, u, v \in \mathbb{F}_q\}$ with

$$x_S(t, u, v)\, x_S(t', u', v') = x_S(t+t', u+u'-t(t')^{3\theta}, v+v'-t'u + t(t')^{3\theta+1} - t^2(t')^{\theta})$$

Cor $x_S(t, u, v)^{x_S(t', u', v')} = x_S(t, u - t(t')^{3\theta} + t't^{3\theta}, v - t'u + tu' + t(t')^{3\theta+1} - t't^{3\theta+1} - t^2(t')^{3\theta} + (t')^2 t^{3\theta})$

$n_r(t) = x_r(t) x_{-r}(t^{-1}) x_r(t)$ $h_r(t) = n_r(t) n_r(-1)$ $h(t) = h_a(t) h_b(t^{3\theta})$

$H = \{h(s) \mid s \in \mathbb{F}_q^\times\}$

$x_S(t, u, v)^{h(s)} = x_S(s^{3\theta-2} t, s^{1-3\theta} u, s^{-1} v)$

$B = QH$ is a Borel subgroup of G_0,

$G = B \sqcup BgB$ for some $g \notin B$

$Z(Q) \cong \{x_S(0, 0, v) \mid v \in \mathbb{F}_q\}$

$Q' = \{x_S(0, u, v) \mid u, v \in \mathbb{F}_q\}$ is an elementary abelian 3-group of order q^2

$Q \setminus Q' = \{$all elements of order 9 in $Q\}$

[G_0 - classes of unipotent elements].

Let $z = X_S(0,0,1) \in Z(Q)$
$y = X_S(0,1,0) \in Q' \setminus Z(Q)$
$x = X_S(1,1,0) \in Q \setminus Q'$
$x' \in Q \setminus Q'$, and (x') is not conjugate to $(x')^{-1}$ in G_0 $\in (G_0)_W \cong {}^2G_2(q_0)$

Lemma: $\boxed{z, \ y, \ y^{-1}, \ x}$, (x'), $(x')^{-1}$ is a set of representative for G_0 - class of unipotent elements.

[pf]: Assume $x_1, x_2 \in Q$, and $g \in G_0$
$x_1^g = x_2$, $\Rightarrow g \in B$.

(Suitable for twisted Lie rank 1 simple group)

$x = X_S(t, u, v)$ $(t \in \mathbb{F}_q^\times)$ is conjugate to x^{-1} in G_0
$\Leftrightarrow$ there exists $t' \in \mathbb{F}_q$ such that
$u - t^{3\theta+1} = -(t')^{3\theta} t + t^{3\theta} t'$

TABLE 22.2.7 **Unipotent classes and centralizers in** ${}^2G_2(q)$, $q = 3^{2n+1}$

(Unipotent elements and nilpotent elements in simple algebraic group)

class rep. in G	no. of G_σ - class in $u^G \cap G_\sigma$	centralizer order in G_σ
1	1	$\lvert {}^2G_2(q) \rvert$
$(\tilde{A}_1)_3$	1	q^3
G_2	3	$3q, \ 3q, \ 3q$
$G_2(a_1)$	2	$2q^2, \ 2q^2$

[Subgroups].

G_0 has four cyclic Hall subgroup

i) $A_0 \cong C_{\frac{q-1}{2}}$ $\quad N_{G_0}(A_0) \cong D_{2(q-1)}$

ii) $A_1 \cong C_{\frac{q+1}{4}}$ $\quad N_{G_0}(A_1) \cong (C_2^2 \times C_{\frac{q+1}{4}}) : C_6$

iii) $A_2 \cong C_{q-\sqrt{3q}+1}$ $\quad N_{G_0}(A_2) \cong C_{q-\sqrt{3q}+1} : C_6$

iv) $A_3 \cong C_{q+\sqrt{3q}+1}$ $\quad N_{G_0}(A_3) \cong C_{q+\sqrt{3q}+1} : C_6$

Let S_2 be a Sylow 2-subgroup of G_0, and S_3 be a Sylow 3-subgroup of G_0, respectively

v) $N_{G_0}(S_2) \cong C_2^3 : C_7 : C_3 \cong A\Gamma L_1(8)$

vi) $N_{G_0}(S_3)$ is a Borel subgroup of G_0

LEMMA. (LEMMA 2 of Levchuk&5

A solvable subgroup of G_0 is conjugate to one of the groups in i) $\rightarrow$ (vi)

LEMMA (LEMMA 6 of Levchuk&5

If $K \leq G_0$ is non-solvable. then K is isomorphic to one of the follong four subgroups

i) $PSL_2(8)$ ii) $C_2 \times PSL_2(q')$. iii) $PSL_2(q')$ iv) ${}^2G_2(q')$

Assume G_ω is a maximal subfield subgroup with $\underline{(G_0)_\omega \cong {}^2G_2(q_0)}$, where $q = q_0^r$, and r is an odd prime.

- Characterize fixers $K \leq G$ with $|K| \geq |G_\omega|$.

Lemma. **[Reduction].**

$K_0 := K \cap G_0$ is a fixer of G_0 with $|K_0| \geq |(G_0)_\omega|$

First, we claim that K is solvable.

Assume the contrary K is non-solvable, K_0 is isomorphic to one of i) $PSL_2(8)$. $C_2 \times PSL_2(q')$, $PSL_2(q')$ ${}^2G_2(q')$.

Since $|K_0| \geq \underbrace{|(G_0)_\omega|}_{\cong {}^2G_2(q_0)} = q_0(q_0-1)(q_0^3+1)$, it follows that

K_0 is isomorphic to one of $C_2 \times PSL_2(q')$, $PSL_2(q')$, ${}^2G_2(q')$, with $q' > q_0$.

Then K_0 contains an element of order $\frac{q'-1}{2}$, but $(G_0)_\omega$ does not contain an element of such order. a contradiction.

Thus, K is solvable, and K_0 is conjugate to one of $N_{G_0}(A_i)$, $i \in \{0, 1, 2, 3\}$. $N_{G_0}(S_2)$, $\underline{N_{G_0}(S_3)}$

$\vdots$

Lemma

$K_0 := K \cap G_0$ is conjugate to a subgroup of B.

$q = q_0^r$, r is an odd prime

We may assume $G_w = C_G(\phi^r) = C_{G_0}(\phi^r) = \langle \phi \rangle$

Lemma: K_0 is further conjugate to a subgroup of $QH(q_0)$, where $H(q_0) = \{ h(s) \mid s \in \mathbb{F}_{q_0}^\times \}$

Ref: Matrix generators. (2012).

[pf]: There is a 7-dimensional irreducible representation of G_0: $\rho : G_0 \longrightarrow GL_7(q)$ such that

$\rho(B) = \{$upper-triangular matrices in $\rho(G_0)\}$

$\rho(H) = \{$diagonal matrices in $\rho(G_0)\}$

$\rho((G_0)_w) = \{$all matrices with entries in $\mathbb{F}_{q_0}$ in $\rho(G_0)\}$

$= GL_7(q_0) \cap \rho(G_0)$

Note that ~~the~~ matrices in $\rho(B)$ that are similar to some matrices $GL_7(q_0) \geqslant \rho((G_0)_w)$ are those with diagonal entries in $\mathbb{F}_{q_0}$ $= \rho(QH(q_0))$

"similar means conjugate in $GL_7(q)$."

$\boxed{K_0 \text{ is conjugate to a subgroup of } QH(q_0)}$

$\downarrow$

$H(q_0) \leq (G_0)_w$

Analyze G_0-classes of unipotent elements in $(G_0)_w$.

Does $\underline{(G_0)_w}$ contain an element of order q x, such that x is not conjugate to x^{-1} in $G_0 = {}^2G_2(q)$.

$x = X_S(t, u, v)$ $(t \in \mathbb{F}_q^\times)$ is conjugate to x^{-1} in G_0

$\Leftrightarrow$ there exists $t' \in \mathbb{F}_q$ such that

$u - t^{3\theta+1} = -(t')^{3\theta} t + t^{3\theta} t'$

$(G_0)_w \cong {}^2G_2(q_0) = C_{G_0}(\phi^r)$

Note that each element of order q in $(G_0)_w$ is conjugate to $\underline{X_S(1, u, v)}$ in $(G_0)_w$.

$X_S(1, u, v)$ is conjugate to $X_S(1, u, v)$

$\Leftrightarrow$ there exists $t' \in \mathbb{F}_q$ such that

$\underline{u' - 1} = -(t')^{3\theta} + t'$

By additive Hilbert 90.

$\Leftrightarrow \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_3}(u'-1) = 0$

If $q = q_0^3$, for each $u' \in \mathbb{F}_{q_0}$. each element of ~~order~~ order q in $(G_0)_w$ is conjugate to its inverse.

$$\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_3}(u'-1) = 0.$$
$$\mathrm{Tr}_{\mathbb{F}_{q_0}/\mathbb{F}_3}\left(\underline{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_{q_0}}(u-1)}\right) =$$
$$\mathrm{Tr}_{\mathbb{F}_{q_0}/\mathbb{F}_3}\left([\mathbb{F}_q : \mathbb{F}_{q_0}](u-1)\right) = 0.$$

If $q \neq q_0^3$, there exists u' such that $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_3}(u'-1) \neq 0$, and there exists $x \in (G_0)_w$ such that $o(x) = q$, and x is not conjugate to x^{-1} in G_0.

If $q \neq q_0^3$. each element in Q is conjugate to an element in $(G_0)_w$. $\underline{QH(q_0)}$ is indeed a fixer.

If $q = q_0^3$ each element of order q in $\underline{\dfrac{(G_0)_w}{K \cap Q}}$ is conjugate to its inverse.

$K_0{}^g \leq QH(q_0)$ for some $g \in G_0$.

We might assume $K_0 \leq QH(q_0)$ (since K^g is still a fixer).

$x = X_S(t, \underbrace{u, v)}_{\in K \cap Q}$ $(t \in \mathbb{F}_q^\times)$ is conjugate to x^{-1} in G_0

$\Leftrightarrow$ there exists $t' \in \mathbb{F}_q$ such that

$$u - t^{3\theta+1} = -(t')^{3\theta} t + t^{3\theta} t'$$

$$u = t^{3\theta+1}\Big(\underbrace{\frac{t'}{t} - \Big(\frac{t'}{t}\Big)^{3\theta}}_{\in A} + \underbrace{1}_{\in A}\Big)$$

$A = \{ \alpha - \alpha^{3\theta} \mid \alpha \in \mathbb{F}_q \}$ = kernel of trace map from $\mathbb{F}_q$ to $\mathbb{F}_3$

$Tr_{\mathbb{F}_q / \mathbb{F}_3}(1) = \underline{[\mathbb{F}_q : \mathbb{F}_3]} = 0 . \Rightarrow 1 \in A$

$x = X_S(t, u, v)$ is conjugate to $x^{-1} \Leftrightarrow \underline{u \in t^{3\theta+1} A}$.

$= \{ t^{3\theta+1} \alpha \mid \alpha \in A \}$

$K \cap Q' / K \cap Z(Q) = \{ u \in \mathbb{F}_q \mid$ there exists $v \in \mathbb{F}_q$ such that $X_S(0, u, v) \in K \}$

$K \cap Q / K \cap Q' = \{ t \in \mathbb{F}_q \mid$ there exists $u, v \in \mathbb{F}_q$ such that $X_S(t, u, v) \in K \}$.

$$\boxed{\text{Goal : } |K \cap Q| \leq q^2}$$

Claim : For any $u \in K \cap Q' / K \cap Z(Q)$

$$u \in \prod_{t \in K \cap Q / K \cap Q'} t^{3\theta + 1} A.$$

There exists v_1, $u_2, v_2 \in \mathbb{F}_q$. $X_S(0, u, v_1) \in K$, $X_S(t, u_2, v_2) \in K$

$$X_S(t, u_2, v_2) \cdot X_S(0, u, v_1) = X_S(t, u_2 + u, v_2 + v_1) \in K.$$

$u_2 \in t^{3\theta+1} A$, $u_2 + u \in t^{3\theta+1} A$. $\Rightarrow u \in t^{3\theta+1} A$.

- Study $\prod_{t \in T} tA$ for $T \subseteq \mathbb{F}_q$ and its size.

• Study $\bigcap_{t\in T} tA$ for $\underline{T\subseteq \mathbb{F}_q}$ and its size.

Lem: Let $\mathrm{span}_{\mathbb{F}_3}(T^{-1})$ be the additive subgroup of $\mathbb{F}_q$ generated by $\{t^{-1} \mid t\in T\}$

$$\mathrm{span}_{\mathbb{F}_3}(T^{-1}) = \{\sum_i a_i t_i^{-1} \mid t_i \in T\}.$$

$$\bigcap_{t\in T} tA = \bigcap_{\lambda\in \mathrm{span}_{\mathbb{F}_3}(T^{-1})} \frac{1}{\lambda}A. \quad (*)$$

[pf]: By induction on $|T|$.

$|T|=1$, $(*)$ ✓

$|T|=2$, $T=\{t, t_1\}$, if $t_2=-t$ $(*)$ ✓

We may assume $t_2 \neq -t$.

want to show $\underline{tA\cap t_1A \subseteq \frac{1}{\frac{a}{t}+\frac{b}{t_1}}A}$ for any $a, b\in\mathbb{F}_3$.

$$tA\cap t_1A = \{t(\alpha-\alpha^{3\theta}) \mid \alpha\in\mathbb{F}_q\}\cap t_1A$$

$$= \{t_1(\underbrace{\tfrac{t}{t_1}\alpha - (\tfrac{t}{t_1}\alpha)^{3\theta}}_{\in A} + \underline{((\tfrac{t}{t_1})^{3\theta} - \tfrac{t}{t_1})\alpha^{3\theta}}) \mid \alpha\in\mathbb{F}_q\}\cap t_1A$$

$$= \{t(\alpha-\alpha^{3\theta}) \mid \alpha^{3\theta}\in \frac{A}{(\frac{t}{t_1})^{3\theta}-\frac{t}{t_1}}\} \quad ☆$$

$$tA\cap \frac{1}{\frac{a}{t}+\frac{b}{t_1}}A = \{t(\alpha-\alpha^{3\theta}) \mid \alpha^{3\theta}\in \frac{A}{t^{3\theta}(\frac{a}{t}+\frac{b}{t_1})^{3\theta} - t(\frac{a}{t}+\frac{b}{t_1})}\}$$

$$= \{t(\alpha-\alpha^{3\theta}) \mid \alpha^{3\theta}\in \frac{A}{b((\frac{t}{t_1})^{3\theta}-\frac{t}{t_1})}\}$$

$= tA\cap t_1A$. $\boxed{\text{Therefore, } tA\cap t_1A \subseteq \frac{1}{\frac{a}{t}+\frac{b}{t_1}}A}$

$$\mathrm{span}_{\mathbb{F}_3}(T^{-1}) = \{ \sum_i a_i t_i^{-1} \mid t_i \in T \}.$$

$$\bigcap_{t \in T} tA = \bigcap_{\lambda \in \mathrm{Span}_{\mathbb{F}_3}(T^{-1})} \frac{1}{\lambda} A. \qquad (*)$$

Assure $(*)$ holds for $|T| = 1, \cdots k-1$.

we prove it for $|T| = k$.

Then $T = \{t_1, \cdots t_k\}$, for each $\lambda \in \mathrm{Span}_{\mathbb{F}_3}\{t_1^{-1}, \cdots t_k^{-1}\}$

we need to show $\bigcap_{t \in T} tA \subseteq \frac{1}{\lambda} A$

Assure $\lambda \in \mathrm{span}_{\mathbb{F}_3}\{t_2^{-1}, \cdots t_k^{-1}\}$, then by induction

$$\bigcap_{t \in T} tA \subseteq \underline{\bigcap_{i=2}^{k} t_i A} \subseteq \frac{1}{\lambda} A.$$

We may assure $\lambda \notin \mathrm{span}_{\mathbb{F}_3}\{t_2^{-1}, \cdots t_k^{-1}\}$. then there exists $0 \neq a_1, \cdots a_k \in \mathbb{F}_3$ such that $\lambda = a_1 t_1^{-1} + \cdots + a_k t_k^{-1}$

$$\bigcap_{t \in T} tA = t_1 A \cap \left(\bigcap_{i=2}^{k-1} t_i A\right) \subseteq t_1 A \cap \frac{1}{a_2 t_2^{-1} + \cdots + a_k t_k^{-1}} A$$

(by induction again) $$\subseteq \frac{1}{a_1 t_1^{-1} + a_2 t_2^{-1} + \cdots + a_k t_k^{-1}} A.$$

$$= \frac{1}{\lambda} A$$

$\square$

$$\bigcap_{t\in T} tA = \bigcap_{\lambda\in \mathrm{Span}_{\mathbb{F}_3}(T^{-1})} \frac{1}{\lambda}A$$

Lemma:

$$\dim_{\mathbb{F}_3} \bigcap_{t\in T} tA + \dim_{\mathbb{F}_3}(\mathrm{Span}_{\mathbb{F}_3}(T^{-1})) = \dim_{\mathbb{F}_3}(\mathbb{F}_q)$$
$$= 2n+1$$

Claim: For any $u \in K\cap Q'/K\cap Z(Q)$

$$u \in \bigcap_{t\in K\cap Q/K\cap Q'} t^{3\theta+1}A.$$

$$\Rightarrow \dim_{\mathbb{F}_3}(K\cap Q/K\cap Q') + \dim_{\mathbb{F}_3}(K\cap Q'/K\cap Z(Q))$$
$$\leq 2n+1$$

$$\Rightarrow |K\cap Q| \leq q\cdot q = q^2 \qquad |K_0| \leq q^2\cdot(q_0-1) < |(G_0)_\omega|$$

a contradiction

$q \neq q_0{}^3$.

If $(G_0) \cong {}^2G_2(q_0)$, then K_0 is conjugate to a subgroup of $QH(q_0)$, $q\neq q_0{}^3$

Lemma: 1. $\underline{QH(q_0) = \langle \phi \rangle}$ is a fixer.

2. Each fixer is conjugate to a subgroup $\underline{QH(q_0) = \langle \phi \rangle}$.

- Charaterize fixers K with $|K| \geqslant |G_w|$.
  such that K is not conjugate to a subgroup of $P_1 \cdot P_2$.