Some progress on locally-primitive generalized polygons

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joint work with Caiheng Li, Peice Hua

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Some definitions on finite geometries

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$\mathcal{S}$ is a **partial linear space** if any two collinear points are incident with a unique line.
$P \sim Q \Rightarrow L := PQ$, $L \sim M \Rightarrow P' := L \cap M$

Definitions on generalized polygons

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a finite geometry.
The **incidence graph** Γ of $\mathcal{S}$ is the graph with the vertex set $\mathcal{P} \cup \mathcal{L}$ and flags of $\mathcal{S}$ as edges.

Definition

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$\mathcal{S}$ is a **finite generalized *n*-gon** if the following two axioms are satisfied:

- $\mathcal{S}$ contains two ordinary k-gon as a sub-geometry, for $2 \leq k < n$.
- Any two elements $x, y \in \mathcal{P} \cup \mathcal{L}$ are contained in some ordinary n-gon as a sub-geometry in $\mathcal{S}$.

Definitions on generalized polygons

Let $\mathcal{S}$ be a finite generalized n-gon.
$\mathcal{S}$ has order (s, t) if Γ is bi-regular of degree $(s+1, t+1)$, where $s+1$ is the degree of vertices in $\mathcal{L}$.
A finite generalized n-gon $\mathcal{S}$ is called **thick** if every vertex of Γ has degree at least 3.

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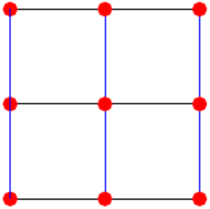
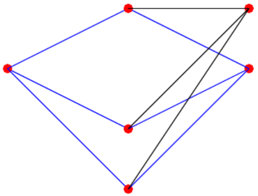
Lemma

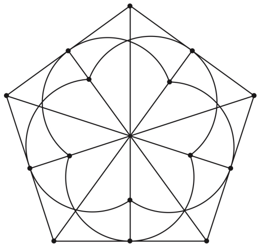
- *(Feit-Higman, 1964) Finite thick generalized n-gons exist only for* $n \in \{3, 4, 6, 8\}$.
- *(V Maldeghem, Generalized Polygons) Every thick generalized n-gon has an order* (s, t) *with* $s, t \geq 2$*; if* $n = 3$ *then* $s = t$*, and if* $n = 8$ *then* $s \neq t$.

Examples of finite generalized n-gons

$n = 4$, examples of thin and thick generalized quadrangles.







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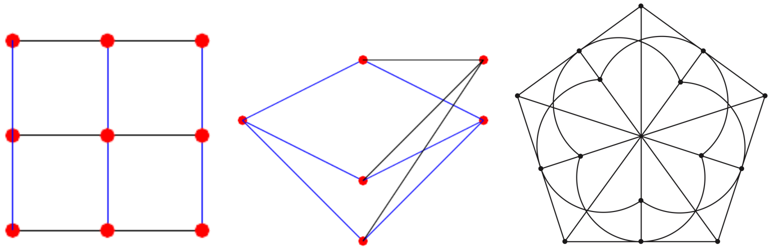
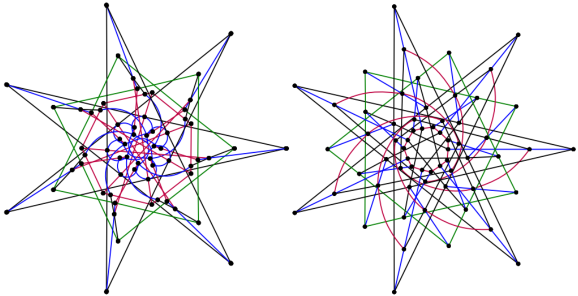


$n = 6$, example of the smallest thick generalized hexagon.



Equivalent definitions on thick n-gons for $n \in \{4, 6, 8\}$

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a finite partial linear space. Then $\mathcal{S}$ is a thick generalized n-gon of order (s, t) if the following two conditions hold:

1. each point is incident with $t + 1$ lines and each line is incident with $s + 1$ points;
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   - $(n = 8)$ 8 is the smallest positive integer k such that $\mathcal{S}$ has a circuit consisting of k points and k lines, and any two elements $x, y \in \mathcal{P} \cup \mathcal{L}$ are contained in some circuit.

Known examples of finite classical thick GQs

The classical GQs, embedded in $PG(d,q)$, $3 \le d \le 5$.

Table: The classical GQs

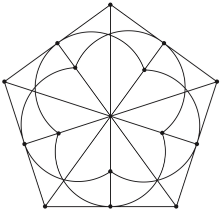
$\mathcal{Q}$	Order	Aut($\mathcal{Q}$)	Symmetry
$W_3(q)$	(q,q)	$P\Gamma Sp_4(q)$	flag-trans., point-prim., line-prim.
$Q_4(q)$, q odd	(q,q)	$P\Gamma O_5(q)$	flag-trans., point-prim., line-prim.
$H_3(q^2)$	(q^2,q)	$P\Gamma U_4(q)$	flag-trans., point-prim., line-prim.
$Q_5^-(q)$	(q,q^2)	$P\Gamma O_6^-(q)$	flag-trans., point-prim., line-prim.
$H_4(q^2)$	(q^2,q^3)	$P\Gamma U_5(q)$	flag-trans., point-prim., line-prim.
$H_4(q^2)^D$	(q^3,q^2)	$P\Gamma U_5(q)$	flag-trans., point-prim., line-prim.

The smallest thick GQ: $GQ(2,2) \cong W_3(2)$

Let $V = \mathbb{F}_q^3$.

- Take $b(x,y) = x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3$;
- $\mathcal{P}$: 1-dimensional isotropic subspace of $V_4(\mathbb{F}_q)$;
- $\mathcal{L}$: 2-dimensional isotropic subspace of $V_4(\mathbb{F}_q)$;
- each 2-dim. isotropic subspace contains $q+1$ points.
- each 1-dim. isotropic subspace is contained in $\frac{\frac{q^3-1}{q-1}-1}{q} = q+1$ lines.
- $x \in \mathcal{P}$,$\ell \in \mathcal{L}$, $x \notin \ell$. $x^\perp \cap \ell$ is a 1-dimensional isotropic subspace.

The unique smallest thick GQ is $W_3(2)$.



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- The GQs AS(q) arose from affine 3-space $AG(3,q)$ with order $(q-1,q+1)$, q is any odd prime power.
- To each regular point P of a GQ $\mathcal{S}$ of order s, there is associated a GQ of order $(s-1,s+1)$, the only examples where s is not a prime power.

For more details, see [S E Payne, J A Thas, Finite Generalized Quadrangles, Chap 3].

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Up to duality, the only known (two infinite families of) examples of **finite generalised hexagons** arose from the finite almost simple groups of Lie type G_2 and ${}^3D_4$, and had orders (q, q) and (q^3, q) respectively.

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Up to duality, the only known (one infinite families of) examples of **finite octagons** arose from the finite almost simple groups of Lie type ${}^2F_4$, and had order $(2^e, 2^{2e})$ with e odd.

For more details, see [H V Maldeghem, Generalized Polygons, Chap 2].

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All known GHs and GOs are flag-transitive, point-primitive, and line-primitive.
All known classical GQs are flag-transitive, point-primitive, and line primitive.

Classify GPs with certain symmetry?

Open problem 1: Classify all flag-transitive finite generalized n-gons?

Conjecture:

- $n = 4$, (Kantor, 1991) up to duality, a finite flag-transitive GQ is classical, or $T_2^*(O)$ arise from hyper-oval O in $PG(2, 2^2)$ and $PG(2, 2^4)$ with order $(3, 5)$ and $(15, 17)$ respectively.

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Furthermore, Are all finite GHs and GOs known?

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problem 2: Classify all point-primitive (and line-primitive) finite generalized n-gons?

Local conditions on generalized n-gons

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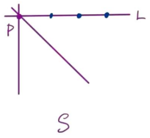
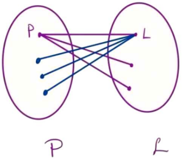
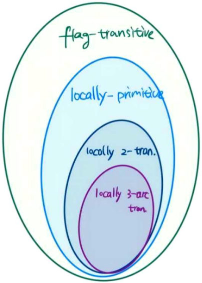
$\mathcal{S}$ is G-**locally transitive** if G_u acts transitively on $\Gamma(u)$ for each u in Γ.
$\mathcal{S}$ is G-**locally primitive** if G_u acts primitively on $\Gamma(u)$ for each u in Γ.
$\mathcal{S}$ is G-**locally** 2-**transitive** if G_u acts 2-transitively on $\Gamma(u)$ for each u in Γ.
$\mathcal{S}$ is G-**locally** 3-**arc transitive** if G_u acts 3-arc transitively on $\Gamma(u)$ for each u in Γ.

Relations between local conditions and integral conditions







flag transitive $\Leftrightarrow$ locally transitive

locally 2-transitive $\Longleftrightarrow \begin{cases} G_p \curvearrowright \Gamma(p) & \text{2-transitive} \\ G_L \curvearrowright \Gamma(L) & \text{2-transitive} \end{cases}$

$\Longleftrightarrow \begin{cases} G_p \curvearrowright \Gamma_2(p) & \text{transitive} \\ G_L \curvearrowright \Gamma_2(L) & \text{transitive} \end{cases}$

$\Longleftrightarrow$ locally 2-arc transitive

locally 3-arc transitive $\Longleftrightarrow \begin{cases} G_p \curvearrowright \Gamma_3(p) & \text{transitive} \\ G_L \curvearrowright \Gamma_3(L) & \text{transitive} \end{cases} \overset{\text{for } n=4}{\Longleftrightarrow}$ antiflag-transitive

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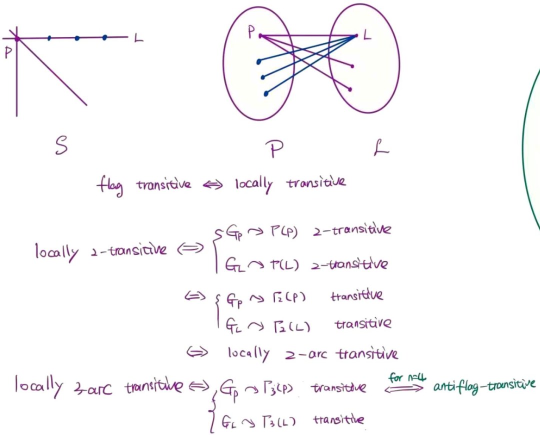

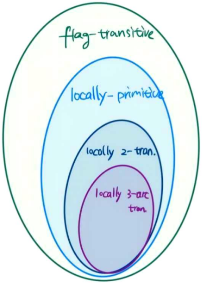

$\mathcal{S}$ is flag-transitive $\Leftrightarrow \mathcal{S}$ is locally transitive $\Leftrightarrow$ Γ is edge-transitive

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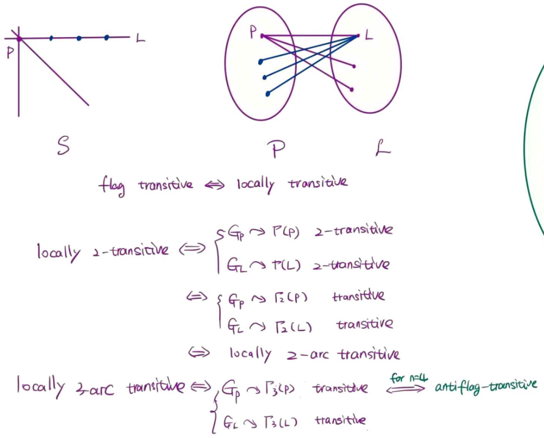

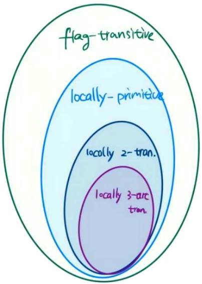

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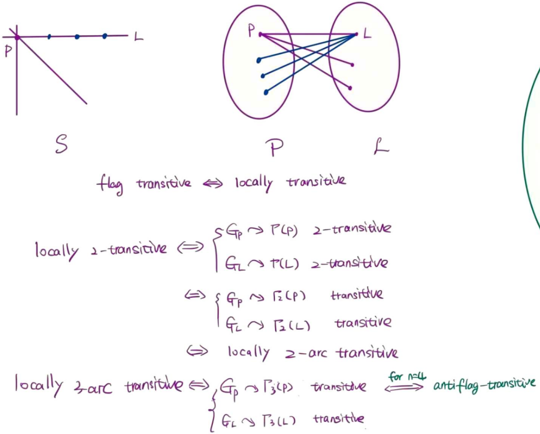
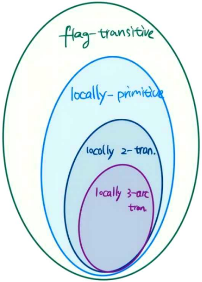
Relations between local conditions and integral conditions





$\mathcal{S}$ is flag-transitive $\Leftrightarrow \mathcal{S}$ is locally transitive $\Leftrightarrow$ Γ is edge-transitive
$\mathcal{S}$ is locally 2-transitive $\Leftrightarrow$ Γ is locally 2-arc transitive.
For $n = 4$, $\mathcal{S}$ is locally 3-arc transitive $\Leftrightarrow$ $\mathcal{S}$ is antiflag-transitive.

Progress on GQs with (integral) primitive conditions

For $n = 4$:

- Bamberg-Giudici-Morris-Poyle-Spiga, (*J.Combin. Theory Ser. A, 2012*), point-primitive+line-primitive $\Rightarrow$ AS type;
- Bamberg-Glasby-Popiel-Praeger, (*J. Combin. Des., 2016*), classified GQs with condition point-primitive+line-transitive+HA type
- Bamgerg-Popiel-Praeger,(*J. Group Theory, 2017*), point-primitive+line-transitive $\Rightarrow$ cannot be HS, HC type;
- Bamberg-Popiel-Praeger, (*Nagoya Math. J., 2019*), point-primitive $\Rightarrow$ cannotbe HC type,
- Feng-Di, (*preprint, 2024+*), point-primitive $\Rightarrow$ cannot be HS type
- Feng-Lu, (*preprint, 2024+*), classified GQs with condition point-primitive +line-primitive+ $Soc(G) = PSL_n(q)$

Progress on GQs with local conditions

- Bamberg-Li-Swartz (*Trans. Amer. Math. Soc., 2018*), classified antiflag-transitive (i.e. locally 3-arc transitive) and locally 2-transitive generalized quadrangles.
- Bamberg-Li-Swartz (*Trans. Amer. Math. Soc., 2021*), classified locally 2-transitive generalized quadrangles.

Is it possible to give a classification of locally-primitive GQs?

Progress on GHs and GOs with certain conditions

For $n = 6$ or 8:

- Schneider-Hendrik-Van Maldeghem (*J.Combin. Theory Ser. A, 2008*), G-point-primitive+flag-transitive $\Rightarrow$ G must be an almost simple group of Lie type.
- Bamberg-Glasby-Popiel-Praeger-Schneider(*J.Combin. Theory Ser. A, 2017*), G-point-primitive $\Rightarrow$ G must be an almost simple group of Lie type.

Is it possible to give a classification of locally 2-transitive (locally primitive) GHs or GOs ?

Primitive groups and quasiprimitive groups

Suppose that G acts transitively on Ω.

A **block** of Ω is a nonempty proper subset B of Ω s.t. $B^g = B$ or $B^g \cap B = \emptyset, \forall g \in G$.

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G is a **primitive permutation group** if G acts faithfully and primitively on some set Ω.

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- each minimal normal subgroup is transitive;
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G is a **quasiprimitive permutation group** on Ω if each minimal normal subgroup is transitive on Ω.

Each primitive permutation group is quasiprimitive.

O'Nan-Scott-Praeger type for (quasi-)primitive groups

Abbreviation	O'Nan-Scott type	Descriptions on minmal normal subgroups N
HA	Affine	1; elementary abelian;regular
HS	Holomorph simple	2; non-abelian simple; regular
HC	Holomorph compound	2; both isomorphic to $T^k, k \geq 2$; regular
AS (As)	Almost simple	1;non-abelian simple; non-regular
TW (Tw)	Twisted wreath	1; $N \cong T^k$, $k \geq 6$; regular; G acts transitively on the k simple direct factors of N
SD(Sd)	Simple diagonal	1;$N \cong T^k, k \geq 2$; non-regular; $N_\alpha$is a full diagonal subgroup of N; G acts primitively on the k simple direct factors of N
CD(Cd)	Compound diagonal	1; $N \cong T^{kr}, k, r \geq 2$; non-regular; G acts transitively on the kr direct factors of N; $N_\alpha \cong T^r$ is a direct product of r pair-wise disjoint full strips of length k; the support of any full strip forms a block with minimal size
PA (Pa)	Product action	1; $N \cong T^k, k \geq 2$; non-regular; G acts transitively on the k simple direct factors of N; N_α is a subdirect subgroup of R^k, and $N_\alpha \cong R^k$ for some proper non-trivial subgroup R of T

Table: 8 types of (quasi-)primitive permutation groups

Parameter conditions on Generalized n-gons

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ be a generalized n-gon of order (s, t) for $n \in \{4, 6, 8\}$ with incidence graph Γ.

Lemma (Payne-Thas, Finite Generalized Quadrangles)

For $n = 4$, then

(i) $|\mathcal{P}| = (s+1)(st+1)$, *and* $|\mathcal{L}| = (t+1)(st+1)$;

(ii) $s \leq t^2$, *and* $t \leqslant s^2$;

(iii) $|\Gamma_i(x)| = (t+1)s^{\lfloor \frac{i}{2} \rfloor} t^{\lfloor \frac{i-1}{2} \rfloor}$, $1 \leqslant i \leqslant 3$, *and* $|\Gamma_4(x)| = s^2 t$.

(iv) $s + t \mid st(s+1)(t+1)$.

Lemma (Maldeghem, Generalized Polygons)

For $n = 6$, then

(i) $|\mathcal{P}| = (s+1)(s^2t^2+st+1)$, *and* $|\mathcal{L}| = (t+1)(s^2t^2+st+1)$;

(ii) *st is a perfect square, and $t \leqslant s^3$;*

(iii) $|\Gamma_i(x)| = (t+1)s^{\lfloor \frac{i}{2} \rfloor} t^{\lfloor \frac{i-1}{2} \rfloor}, \ 1 \leqslant i \leqslant 5$, *and* $|\Gamma_6(x)| = s^3t^2$;

(iv) $|\mathcal{P}|_2 = (s+1)_2$, *and* $|\mathcal{L}|_2 = (t+1)_2$.

Lemma (Maldeghem, Generalized Polygons)

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(iv) $|\mathcal{P}|_2 = (s+1)_2$, *and* $|\mathcal{L}|_2 = (t+1)_2$.

For $n = 8$ then

(i) $|\mathcal{P}| = (s+1)(st+1)(s^2t^2 + 1)$, *and* $|\mathcal{L}| = (t+1)(st+1)(s^2t^2 + 1)$;

(ii) *$2st$ is a perfect square, and $t \leqslant s^2$;*

(iii) $|\Gamma_i(x)| = (t+1)s^{\lfloor \frac{i}{2} \rfloor} t^{\lfloor \frac{i-1}{2} \rfloor}$, $1 \leqslant i \leqslant 7$, *and* $|\Gamma_8(x)| = s^4t^3$;

(iv) *$s \neq t$, and st is even. In addition, either $|\mathcal{P}|$ is odd and $|\mathcal{L}|_2 = (t+1)_2$, or $|\mathcal{L}|$ is odd and $|\mathcal{P}|_2 = (s+1)_2$.*

Benson type argument

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$$f_i = |\{P \in \mathcal{P} : d(P, P^g) = 2i\}|,\ g_i = |\{L \in \mathcal{L} : d(L, L^g) = 2i\}|,\ i \leq n/2.$$

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Lemma (Temmermans-Thas-Van Maldeghem, Combinatorica, 2009)

There is $(1+t)f_0 + f_1 = (1+s)g_0 + g_1$. *Moreover,*

(i) *for* $n = 4$,
$(1+t)f_0 + f_1 = k(s+t) + (1+s)(1+t)$ *for some integer* k;

(ii) *for* $n = 6$,
$(1+t)f_0 + f_1 = k_1(s+t+\sqrt{st}) + k_2(s+t-\sqrt{st}) + (1+s)(1+t)$ *for some integer* k_1 *and* k_2;

(iii) *for* $n = 8$,
$(1+t)f_0 + f_1 = k_1(s+t+\sqrt{2st}) + k_2(s+t-\sqrt{2st}) + k_3(s+t) + (1+s)(1+t)$ *for some integer* k_1, k_2 *and* k_3.

The fixed element structure $\mathcal{S}_g$

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a n-gon of order (s, t) for $n \in \{4, 6, 8\}$, with incidence graph Γ.

Let g be any automorphism of $\mathcal{S}$.
Define the fixed element structure $\mathcal{S}_g$ of g to be the subgeometry with points fixed by g, lines fixed by g.

Lemma

Then $\mathcal{S}_g$ is in one of the following 4 cases:

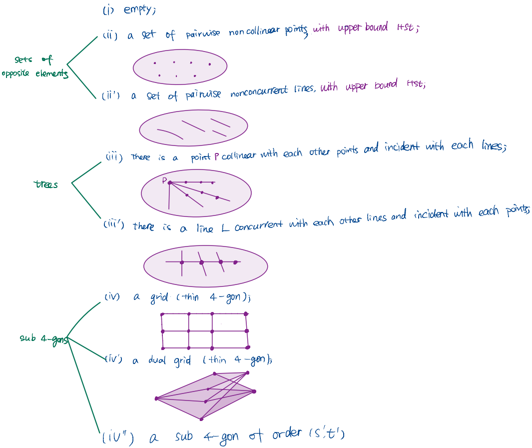
(i) *empty;*

(ii) *consists of a set of elements all opposite one another;*

(iii) *a tree of diameter at most n in the incidence graph Γ;*

(iv) *a sub-n-gon, may be thin.*

Example: $\mathcal{S}_g$ for $n = 4$



Normal quotient graphs

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The normal quotient graph Γ_N of Γ has vertex set the N-orbits on $V\Gamma$, and two N-orbits O_1 and O_2 are adjacent in Γ_N if and only if there exist $x_i \in O_i$ such that x_1 and x_2 are adjacent in Γ.

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Γ is a **cover** of Γ_N if $|\Gamma(x_1) \cap B_2| = 1$ for each edge $\{O_1, O_2\}$ in Γ_N and for each $x_1 \in O_1$.

Lemma (Giudici-Li-Praeger, *Trans. Amer. Math. Soc., 2004*)

Let Γ be a connected G-locally primitive bipartite graph. Let $N \trianglelefteq G$, the maximal subject to being intransitive on the two parts. Then Γ is a cover of Γ_N and N acts semiregularly on the two parts. Moreover, Γ_N is a (G/N)-locally primitive bipartite graph satisfying at least one of the following:

(i) $\Gamma_N \cong K_{m,m'}$;

(ii) *G/N acts faithfully and quasi-primitively on the two parts of on Γ_N;*

(iii) *G/N acts faithfully on the two parts of Γ_N but only acts quasi-primitively on one of them.*

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Lemma (Giudici-Li-Praeger, *Trans. Amer. Math. Soc., 2004*)

Let Γ be a G-locally primitive connected bipartite graph such that G acts faithfully and quasiprimitively on both parts with type $\{X, Y\}$. Then either $X = Y$, or $\{X, Y\} = \{Sd, Pa\}$ or $\{Cd, Pa\}$.

G-Locally primitive generalized n-gons

From now on, suppose that

- $n \in \{4, 6, 8\}$;
- $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a generalized n-gon of order (s, t), with incidence graph Γ;
- $\mathcal{S}$ is G-locally primitive, where $G \leq \mathrm{Aut}(\mathcal{S})$.

Outline

- Case 1. G is neither quasiprimi. on $\mathcal{P}$ nor $\mathcal{L}$.
  - ⇒ gcd$(s,t)=1$; normal quotient graph Γ_N
  - Case 1.1 $\Gamma_N \cong K_{1+s,1+t}$ ⇓ G/N acts primi. on both two parts
    - Case 1.1.1 G_N acts faithfully on both parts. excluded for each $n \in \{4,6,8\}$ by using gcd$(s,t)=1$.
    - Case 1.1.2 G_N acts unfaithfully on $\mathcal{L}_N$. up to duality.
      - $n=6$ or 8, excluded
      - $n=4$, remain HA, HS, HC, TW, AS.
  - Case 1.2 $G_N \curvearrowright (\mathcal{P}_N, \mathcal{L}_N)$ faithful, quasiprim. of type (X,X), (Sd, Pa), or (Cd, Pa)
    - $n=6,8$, excluded except for (As, As)
    - $n=4$, remain (Sd, Pa), (Cd, Pa), (As, As), (Pa, Pa)
  - Case 1.3 $G_N \curvearrowright (\mathcal{P}_N, \mathcal{L}_N)$ faithful, quasiprim. of type $(X, -)$.
    - $n=6,8$ excluded
    - $n=4$, remain (HA, –), (HS, –), (HC, –) (TW, –) (AS, –), (Pa, –)
- Case 2. G is quasiprim. on $\mathcal{P}$ (up to duality)
  - Case 2.1 $G \curvearrowright (\mathcal{P}, \mathcal{L})$ faithful. quasiprimi. of type (X,X), (Sd, Pa), or (Cd, Pa)
    - $n=6,8$, excluded except for **(As, As)**
    - $n=4$, remain (TW, –) (Pa, –)
  - Case 2.2 $G \curvearrowright (\mathcal{P}, \mathcal{L})$ faithful, quasiprimi. of type $(X, -)$.
    - $n=6,8$ excluded
    - $n=4$, remain (TW, –), (AS, –) (Pa, –)

Case 1. G is neither quasi-primitive on $\mathcal{P}$ nor $\mathcal{L}$

There is $N \trianglelefteq G$ intransitive on both $\mathcal{P}$ and $\mathcal{L}$, and we get the normal quotient graph Γ_N, with Γ a cover.

Γ_N is a biregular bipartite graph, with valency $\{s+1, t+1\}$.
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Lemma (Li-Hua-Di)

We have $\gcd(s,t)=1$. *Furthermore,*

- *for* $n=4$, $s+t \mid st+1$, *and* $s+t \leq \gcd(t+1, s-1)\cdot\gcd(t-1, s+1)$;
- *for* $n=6$ *or* 8, $4 \nmid 1+s$ *and* $4 \nmid 1+t$.

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- *for* $n = 4$, $s + t \mid st + 1$, *and* $s + t \leq \gcd(t+1, s-1) \cdot \gcd(t-1, s+1)$;
- *for* $n = 6$ *or* 8, $4 \nmid 1 + s$ *and* $4 \nmid 1 + t$.

Proof.

For any $g \in N$, g is fixed-point-free and fixed-line-free since N is semiregular.
For any line $L \in \mathcal{L}$, the N-orbit L^N is a set of pairwise non-concurrent lines.
$g_1 = 0$, $f_1 = 0$.
By Benson type argument, we have $\gcd(s,t) = 1$.

Case 1.1 $\Gamma_N \cong K_{1+s,1+t}$

Lemma (Fan-Li-Pan, J. Group Theory, 2014)

For a complete bipartite graph $K = K_{m,n}$ with $m, n \geq 3$, a group $H \leq \mathrm{Aut}(K)$ acts on K locally primitively iff H is primitive on both parts, and one of the following three cases holds:

- *H acts faithfully on both parts, and all possible pairs of $\{m, n\}$ are given.*
- *H acts faithfully on one of the two parts, and $K \cong Cos(H, L, R)$, where L, R are two maximal subgroups such that L is core-free, $L \cap R$ is maximal in R, and R contains a nontrivial normal subgroup of H.*
- *H acts unfaithfully on both parts, and $X = (M \times N).P$ such that $M.P$ and $N.P$ are primitive permutation groups of degree n and M, respectively.*

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We first exclude the subcase where G/N acts faithfully on both parts by using parameter conditions induced by $\gcd(s,t) = 1$.

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Suppose that G/N acts unfaithllly on $\mathcal{L}_N$.
Then consider the O'Nan-Scott type X induced by G_N on $\mathcal{P}_N$ case by case.

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Then consider the O'Nan-Scott type X induced by G_N on $\mathcal{P}_N$ case by case.
For $n = 6$ or 8, $4 \nmid 1 + s \Rightarrow X \in \{PA, HA, AS\}$.

Lemma (Li-Hua-Di)

Suppose that G_P acts on $\Gamma(P)$ unfaithfully, and denote by $G_P^{[1]}$ this kernel. Then for any $g \in G_P^{[1]}$, the following holds:

(1) *if $n \in \{4, 6\}$, the fixed element structure $\mathcal{S}_g$ of g is either a star with $t+1$lines or a sub-n-gon with order (s', t), where $s' \geq 1$, and $s't < s$;*

(2) *if $n = 8$, the fixed element structure $\mathcal{S}_g$ of g is a star with $t+1$ lines.*

Proof.

- For any point Q fixed by g, g fixes each lines through it.
- For any line fixed by g, there is a constant number of points fixed by g on it.
- $\mathcal{S}_g$ is either a star with $t+1$ lines or a sub-n-gon with order (s', t). In particular, for $n = 8$, there is no sub 8-gon of order (s, t') or (s', t).

□

Lemma (Li-Hua-Di)

The primitive action induced by G/N on $\mathcal{P}_N$ cannot be of type PA.

Proof.

Suppose for a contrary that the primitive permutation action induced by G/N on $\mathcal{P}_N$ is of type PA. Then $soc(G/N^{\mathcal{P}_N}) = T^k$, $k \geq 2$. Set K to be the kernel of G/N on $\mathcal{L}_N$.

- Identify $\mathcal{P}_N$ as Δ^k, and T acts on Δ irregularly with point stabilizer T_δ. Take $u = (\delta, \cdots, \delta) \in \Delta^k$.
- $T^k \leqslant K$, and $T_\delta^k \leqslant K_u$.
- Take $t \in T_\delta$ such that $|fix_\Delta(t)| > 1$. For each $1 \leq i \leq k$, let $\theta_i \in T_\delta^k$ with the first i entries t and the last $k - i$ entries 1.
- The fixed elements substructure $\mathcal{S}_i$ of the pre-image of θ_i in G is a sub-n-gon with order (s_i, t), and $n \in \{4, 6\}$.
- Since $S_2 \subset S_1 \subset \mathcal{S}$, s is equal to a power of t, which contradicts with $\gcd(s, t) = 1$.

□

Lemma (Li-Hua-Di)

The primitive action induced by G/N on $\mathcal{P}_N$ cannot be of type HA for $n \in \{6, 8\}$.

Proof.

Suppose for a contrary that the primitive permutation action induced by G/N on $\mathcal{P}_N$ is of type HA. Then $soc(G/N^{\mathcal{P}_N}) = Z_p^d$, p odd, $d \geq 1$. Set K to be the kernel of G/N on $\mathcal{L}_N$.

- s is even, t is odd, and $1 + t \equiv 2 \pmod 4$.
- $X := G/N$ on $\mathcal{L}_N$ is of AS type.
- $X = (Z_p^d \times T).P$ or $X \lesssim Z_p^d.(T.Out(T))$ according to G/N acts unfaithfully on $\mathcal{P}_N$ or not.
- If $X = (Z_p^d \times T).P$, choose $t_1, t_2 \in T_\delta$, where $Cos(T : T_\delta) \cong \mathcal{L}_N$. Then $(\mathcal{S}_{\hat{t}_1} \cap \mathcal{S}_{\hat{t}_2}) \subset \mathcal{S}_{\hat{t}_1} \subset \mathcal{S}$, a contradiction.
- If $X \lesssim Z_p^d.(T.Out(T))$, then $T \leq GL_d(p)$, an irreducible subgroup. So $d > 1$.

□

Proof.

- For $n = 6$, and s is a square. By the Catalan's Conjecture, $1 + s = p^d$ has a unique pair of solution: $2^3 + 1 = 3^2$. Excluded.
- For $n = 8$, t is a square, so $|T|_p = |T_\delta|_p$ for any prime $p \equiv 3 \pmod 4$.

□

Lemma

The primitive action induced by G/N on $\mathcal{P}_N$ cannot be of type AS.

Proof.

Suppose that the primitive permutation action induced by G/N on $\mathcal{P}_N$ is of type AS. Then $(G/N^{\mathcal{P}_N}) = T$, with point stabilizer T_u, where $u \in \mathcal{P}_N$.

- T acts trivially on $\mathcal{L}_N$.
- Chose t such that $|fix_{\mathcal{P}_N}(t)| > 1$. Then the fixed element substructure $\mathcal{S}_{\hat{t}}$ is a sub-n-gon with order (s', t), where $t \leq s't < s$ and $n \in \{4, 6\}$.
- G/N on $\mathcal{L}_N$ is also of type AS.
- If the action of G/N on $\mathcal{P}_N$ is unfaithful, we can deduce $s < t$, which cannot occur. So, G/N acts faithfully on $\mathcal{P}_N$.
- T has a transitive permutation representation of degree $t + 1$. So we have $t + 1 \geq p(T)$, where $p(T)$ denotes the least permutation representation of T. $Out(T)$ induced a primitive action of degree $s + 1$. So, $s + 1 \mid |Out(T)|$.
- $(s + 1)^2 \leq |Out(T)|^2 \leq p(T) \leq t + 1$ except for 5 possibilities.

Case 1.1. $G_N \cong K_{1+s,1+t}$

Conclusion:

- For $n = 6$ or 8, **excluded**;
- For $n = 4$, up to duality, it must be that G/N acts unfaithlly on $\mathcal{L}_N$, and induced a primitive action on $\mathcal{P}_N$ of one of type HA, HS, HC, TW, AS.

Case 1.2. G/N acts faithfully and quasiprimitively on both parts

Conclusion:

- For $n = 6$ or 8, the remaining quasiprimitive type is (As, As), or (Pa,Pa);
- For $n = 4$, up to duality, the remaining quasiprimitive type is one of (Sd,Pa), (Cd, Pa), (As, As) and (Pa,Pa).

Case 1.3. G/N acts faithfully on both parts, but quasiprimitively on one of them

Conclusion:

- For $n = 6$ or 8, **excluded**;
- For $n = 4$, up to duality, the remaining quasiprimitive type is one of (HA,-), (HS,-), (HC,-), (TW,-), (As,-).

Case 2. G acts quasiprimitively on $\mathcal{P}$ up to duality

In this case, G acts faithfully on both $\mathcal{P}$ and $\mathcal{L}$.

Conclusion:

- For $n = 6$ or 8, the remaining quasiprimitive type is (As,As), (Pa,Sd), or (Pa,Cd);
- For $n = 4$, up to duality, the remaining quasiprimitive type is one of (Tw,-), (Tw,Tw), (Pa,Pa), (Pa,Sd), (Pa,Cd), (As,As).

Thanks for your attention!