Regular Cayley Maps of Elementary abelian p-groups

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Factorizations of groups

Definition (Factorizations of groups)

- *A group X is said to be properly factorizable if $X = GH$ for two proper subgroups G and H of X.*
- *The expression $X = GH$ for two proper subgroups G and H of X is called a factorization of X, and G, H are called its factors.*
- *We say that X has an exact factorization if $G \cap H = 1$.*

Factorizations of groups

Definition (Product groups)

- *X is the product group of G and H.*
- *Given a group G, a group X is called a skew product group of G if $X = GC$ where $G \cap C = 1$ and C is a cyclic group with trivial core in X.*
- *Given a group G, a group X is called a dihedral-skew product group of G if $X = GD$ where $G \cap D = 1$ and D is a dihedral group with trivial core in X.*
- *Given a group G, a group X is called a dihedral-product group of G if $X = GD$ where $G \cap D = 1$ and D is a dihedral group.*

Factorizations of groups

Factorizations of groups naturally arise from the well-known Frattini's argument, including its version in permutation groups.

Proposition (Frattini's argument)

Let X be a group acting transitively on a set Ω, G be a subgroup of X and X_α be a point stabilizer of X. If G acts transitively on Ω, then $X = GX_\alpha$.

G acts regularly on Ω if $G \cap X_\alpha = 1$. A (dihedral-)skew product group X of G acts faithfully on $[X : C]([X : D])$, and contains a regular subgroup G.

Factorizations of groups

History

- *Jones determined all primitive permutation groups containing a transitive cyclic subgroup in 2004.*
- *Li Caiheng and Praeger extended it to quasi-primitive groups, almost simple groups and innately simple groups in 2012.*
- *Li Caiheng determined all quasiprimitive permutation group containing a dihedral regular subgroup in 2006.*
- *Li Caiheng, Pan Jiangmin, and Xia Binzhou determined all quasiprimitive permutation groups with a metacyclic transitive subgroup in 2021.*

Remark: A permutation group is called *quasiprimitive* if each of its non-trivial normal subgroups is transitive.

Unoriented Regular Cayley Maps

- A map on a surface is a cellular decomposition of a closed surface into 0-cells called vertices, 1-cells called edges and 2-cells called faces. The vertices and edges of a map form its underlying graph. A map is said to be orientable if the supporting surface is orientable, and nonorientable if the supporting surface is nonorientable.
- A map $\mathcal{M}$ is said to be orientable regular or (unoriented) regular if its automorphism group $\mathrm{Aut}(\mathcal{M})$ is regular on arcs or flags (in most cases, arcs are incident vertex-edge pairs and flags are incident vertex-edge-face triples).
- A orientable regular map $\mathcal{M}$ is reflexible if it is also regular, and is chiral if it is not regular.

Unoriented Regular Cayley Maps

Definition (Regular maps)

For a given group X and three involutions t, r, ℓ in X, a quadruple $\mathcal{M} = \mathcal{M}(X; t, r, \ell)$ is called a regular map if they satisfy three conditions: (1) $t\ell = \ell t$; (2) $X = \langle t, r, \ell \rangle$; (3) $\langle t, r \rangle$ is core free in X.

A regular map $\mathcal{M}$ is nonorientable if $X = \langle tr, t\ell \rangle$ and orientable if $|X : \langle tr, t\ell \rangle| = 2$.

Unoriented Regular Cayley Maps

Definition (Orientable regular maps)

For a given group Y*, an involution* τ *in* Y *and* $\rho \in Y$*, a triple* $\mathcal{M} = \mathcal{M}(Y; \tau, \rho)$ *is called an orientable regular map if* $Y = \langle \tau, \rho \rangle$ *and* $\langle \rho \rangle$ *is core free in* Y*.*

An orientable regular map $\mathcal{M} = \mathcal{M}(Y; \tau, \rho)$ is reflexible if there exists $X = \langle t, r, \ell \rangle$ such that $Y = \langle t\ell, \rho = tr \rangle$.

Unoriented Regular Cayley Maps

Richter et al. show that a map $\mathcal{M}$ is a Cayley map of a group G if and only if the group $\mathrm{Aut}(\mathcal{M})$ contains a regular subgroup that is isomorphic to G.

Proposition (Jajcay and Širáň, 2003)

Let $\mathcal{M}$ be an orientable regular Cayley map of G. Then $\mathrm{Aut}(\mathcal{M})$ is a skew product group of G.

Proposition (Kwak and Kwon, 2006)

Let $\mathcal{M}$ be a unoriented regular Cayley map of G. Then $\mathrm{Aut}(\mathcal{M})$ is a dihedral-skew product group of G.

Unoriented Regular Cayley Maps

History (Orientable regular Cayley maps)

- $\mathbb{Z}_n$: *Conder and Tucker, 2014.*
- D_{2n}: *I. Kovács and Y.S. Kwon, 2021.*
- *nonabelian characteristically simple groups: Chen Jiyong, Du Shaofei and Li Caiheng, 2022.*
- $\mathbb{Z}_p^n$: *Du Shaofei, Yu Hao and Luo Wenjuan, 2023.*

History (unoriented regular Cayley maps)

- $\mathbb{Z}_n$: *Hu Kan and Y.S. Kwon, 2024+.*

Let CM be an unoriented regular Cayley map of $G = \mathbb{Z}_p^n$. Then $X = \mathrm{Aut}\,(\mathrm{CM})$ is a dihedral-skew product group of G.

$$\mathrm{CM} = (X; t, r, \ell),\ t\ell = \ell t,$$

$$X = \langle t, r, \ell \rangle = G\langle t, r \rangle,\ G \cap \langle t, r \rangle = 1,\ \langle t, r \rangle_X = 1.$$

Set $D = \langle t, r \rangle \cong D_{2m}$, $P \in Syl_p(X)$ and $\sigma := rt$ with order of m.
A regular map $\mathcal{M}$ is called a *regular p-map* if the number of vertices is p^k, where p is prime and $k \geq 1$. Moreover, a regular p-map $\mathcal{M}$ is *normal* if the Sylow p-subgroup of $\mathrm{Aut}\,(\mathcal{M})$ is normal.
Unoriented regular Cayley maps of $\mathbb{Z}_p^n$ are regular p-maps,

Proposition (Du Shaofei, Tian Yao and Li Xiaogang, 2023)

Let $\mathcal{M} = \mathcal{M}(X; t, r, \ell)$ be a regular p-map and P a Sylow p-subgroup of $\mathrm{Aut}\,(\mathcal{M})$. *Then $\mathcal{M}$ is normal, except for the following two cases:*

(1) $p = 2$, $X/O_2(X) \cong \mathbb{Z}_m \rtimes \mathbb{Z}_2$ *or* $\mathbb{Z}_m \rtimes D_4$, *where $m \geq 3$ is odd. Moreover, if $\mathcal{M}$ is nonoriented, then* $X/O_2(X) \cong \mathbb{Z}_m \rtimes \mathbb{Z}_2$ *and* $\ell \in O_2(X)$.

(2) $p = 3$, $X/O_3(X) \cong S_4$.

Let CM be an unoriented regular Cayley map of $G = \mathbb{Z}_p^n$.
Normal: If $p = 2$, then X is a 2-group.
If p is odd, then $G \leq P$ and $X = (P\langle\sigma\rangle) \rtimes \langle t\rangle = (G\langle\sigma\rangle) \rtimes \langle t\rangle$. $(\sigma = rt)$

$p = 2$

Lemma

Let $H = AD$ be a dihedral-skew product group of a 2-group A. Let t, r, ℓ be different involutions in H such that $t\ell = \ell t$, $D = \langle t, r \rangle$ and $H = \langle t, r, \ell \rangle$. Set $\sigma = rt$ and $\ell = g\sigma^i t^j$ with $g \in A$ and some integers i, j. Suppose that g is an involution in A and $H = \langle gt^{j+1}, \sigma \rangle$. Then H is not a 2-group.

There is no exists any nonorientable normal regular Cayley map of $\mathbb{Z}_2^n$.

$p = 2$

Proof For the contrary, assume that H is a 2-group. Then $H = AD$ is not a dihedral group. H can not be generated by two involutions, which implies $g, t, r \notin \Phi(H)$. Consider $\overline{H} = H/\Phi(H)$. Then $\overline{H}$ is either $\langle \overline{g}, \overline{\sigma} \rangle$ or $\langle \overline{gt}, \overline{\sigma} \rangle$.

Suppose that $\overline{H} = \langle \overline{g}, \overline{\sigma} \rangle$. Then the element $\bar{t}$ is $\overline{g}$, $\overline{\sigma}$ or $\overline{g\sigma}$. If $\bar{t} = \overline{g}$, then $gt \in \Phi(H)$ and so $H = \langle g, t, r \rangle = \langle gt, t, r \rangle = \langle t, r \rangle$ is a dihedral group, a contradiction. If $\bar{t} = \overline{\sigma}$, then $r = \sigma t \in \Phi(H)$, a contradiction. If $\bar{t} = \overline{g\sigma}$, then $tg\sigma \in \Phi(H)$, which implies $H = \langle tg\sigma, t, r \rangle = \langle t, r \rangle$ is a dihedral group, a contradiction again.

Suppose that $\overline{H} = \langle \overline{gt}, \overline{\sigma} \rangle$. Then the element $\overline{g}$ is $\overline{gt}$, $\overline{\sigma}$ or $\overline{gt\sigma}$. With the same argument as the above, we get also a contradiction.

Therefore, H is not a 2-group. □

p is odd

For a map $\mathcal{M} = \mathcal{M}(X; t, r, \ell)$, its Petrie dual $P(\mathcal{M})$ is $\mathcal{M}(X; t, r, \ell t)$.

Lemma

Let $\mathcal{M}$ be a normal regular p-map. If $\mathcal{M}$ is nonorientable(orientable), then its Petrie dual $P(\mathcal{M})$ is orientable(nonorientable), separately.

Using the above lemma, we can get that for any nonorientable regular Cayley map CM of $G \cong \mathbb{Z}_p^n$, if p is odd and CM is normal, then $P(\mathcal{M})$ is orientable. Since $P(\mathrm{CM})$ is also a regular Cayley map of $G \cong \mathbb{Z}_p^n$, it is a reflexible Cayley map of $G \cong \mathbb{Z}_p^n$. With the help of results about orientable regular Cayley maps, we shall classify nonorientable regular Cayley maps of $G \cong \mathbb{Z}_p^n$ with odd prime p.

Proposition (Du, Yu, Luo, 2023)

Every orientable regular Cayley map of $G \cong \mathbb{Z}_p^n$ is isomorphic to $\mathcal{M}(Y;\sigma, g(-I))$, where Y, σ and g meet the following:

(1) $Y = T \rtimes \langle\sigma\rangle \leq \mathrm{AGL}(n,p)$, *where* $T \cong \mathbb{Z}_p^n$, $|T \cap G| \geq p^{n-1}$; *and* $T = G$ *if* $p = 2$;

(2) $\sigma \in \mathbf{M}(n,p)$;

(3) g *may be taken from* T *such that* $T = \langle g, g^{\sigma}, \cdots, g^{\sigma^{n-1}}\rangle$.

Moreover, given p *and* n, *different* σ *in* $\mathbf{M}(n,p)$ *give nonisomorphic maps.*

p is odd

Lemma

Let $\mathrm{CM} = \mathcal{M}(X : t, r, \ell)$ *be a nonorientable regular Cayley map of* $G \cong \mathbb{Z}_p^n$. *If* p *is odd and* CM *is normal, then the following holds:*

(1) $n \geq 2$;

(2) $X = \mathrm{Aut}\,(\mathrm{CM}) = T \rtimes \langle t, r \rangle \leq \mathrm{AGL}(n, p)$, *where* $T \cong \mathbb{Z}_p^n$ *and* $|T \cap G| \geq p^{n-1}$;

(3) $\ell = g(-I)$, $\sigma := rt \in \mathbb{M}(n, p)$, $g^t = g$ *and* $\sigma^t = \sigma^{-1}$, *where if we identify* T *with* $V(n, p)$, *then* g *may be taken from* T *such that* $\{\sigma^i(t) \mid 0 \leq i \leq n-1\}$ *is a base for* T.

not normal

Let $\mathrm{CM} = \mathcal{M}(X; t, r, \ell)$ be a nonorientable regular Cayley map of $G \cong \mathbb{Z}_2^n$.

Lemma

$P(\mathrm{CM})$ is a reflexible Cayley map.

Lemma

There is no exists a nonorientable regular Cayley map $\mathrm{CM} = \mathcal{M}(X; t, r, \ell)$ of $\mathbb{Z}_3^n$ such that $X/O_3(X) \cong S_4$.

Theorem (Yu Hao, 2024+)

Let CM be a nonorientable regular Cayley map of $G \cong \mathbb{Z}_p^n$. Then $n \geq 2$, $\mathrm{CM} = \mathcal{M}(T \rtimes \langle t, \sigma t\rangle; t, \sigma t, g(-I))$, where $\sigma \in \mathbb{M}(n, p)$, $T = \langle g, g^{\sigma}, \cdots, g^{\sigma^{n-1}}\rangle$, $g^t = g$ and $\sigma^t = \sigma^{-1}$. Moreover, given p and n, different σ in $\mathbf{M}(n, p)$ give nonisomorphic maps.

Simple group

Theorem (Yu Hao, 2024+)

Let $X = GD$ be a dihedral-skew product group of a nonabelian simple group G. Then either $G \lhd X$ or the triple (X, G, D) is given in Table 1.

Table: Dihedral-skew product groups of G

Row	X	G	D
1	$\mathrm{AGL}(3,2)$	$GL(3,2)$	D_8
2	M_{12}	M_{11}	D_{12}
3	M_{24}	M_{23}	D_{24}
4	$A_{4m}, m \geq 2$	A_{4m-1}	D_{4m}
5	$PGL(2,11)$	A_5	D_{22}
6	$A_{2m+3} \rtimes \mathbb{Z}_2, m \geq 2$	A_{2m+2}	$D_{2(2m+3)}$
7	$\mathrm{Aut}\,(M_{12})$	M_{11}	D_{24}
8	$A_{4m} \rtimes \mathbb{Z}_2, m \geq 2$	A_{4m-1}	D_{8m}

Future research

Determine dihedral-skew product group of all finite nonabelian characteristically simple groups and classify regular Cayley maps of finite nonabelian characteristically simple groups.

Thanks!