Finite linear groups with exactly 3 orbits

Hanyue Yi

Southern University of Science and Technology

2024.04.10

joint work with C.H. Li and Y.Z. Zhu

Actions of linear groups

$V = \mathbb{F}_p^n$ and $G \leqslant \mathrm{GL}(V)$ is a **linear group** on V.

Actions of linear groups

$V = \mathbb{F}_p^n$ and $G \leqslant \mathrm{GL}(V)$ is a **linear group** on V.

- $\mathrm{GL}(V)$ and $\mathrm{SL}(V)$ are **transitive linear groups**, i.e., transitive on $V \setminus \{0\}$.

Actions of linear groups

$V = \mathbb{F}_p^n$ and $G \leqslant \mathrm{GL}(V)$ is a **linear group** on V.

- $\mathrm{GL}(V)$ and $\mathrm{SL}(V)$ are **transitive linear groups**, i.e., transitive on $V \setminus \{0\}$.
- $\mathrm{Sp}(V)$ is also a transitive linear group.

Actions of linear groups

$V = \mathbb{F}_p^n$ and $G \leqslant \mathrm{GL}(V)$ is a **linear group** on V.

- $\mathrm{GL}(V)$ and $\mathrm{SL}(V)$ are **transitive linear groups**, i.e., transitive on $V \setminus \{0\}$.
- $\mathrm{Sp}(V)$ is also a transitive linear group.
- $\mathrm{GU}(V)$ and $\mathrm{O}(V)$ are generally not transitive. e.g. $\mathrm{GU}_3(5)$ has 6 orbits on $\mathbb{F}_{25}^3$.

Researches on linear groups

- Classification of transitive linear groups
  Huppert, 1957 and Hering, 1985

Researches on linear groups

- Classification of transitive linear groups
  Huppert, 1957 and Hering, 1985
- Classification of $\frac{1}{2}$-transitive linear groups
  Liebeck, Praeger and Saxl, 2018
- Linear groups transitive on "special" (totally isotropic, nondegenerate...) subspaces
  Giudici, Glasby and Praeger, 2023

- Classification of transitive linear groups
  Huppert, 1957 and Hering, 1985
- Classification of $\frac{1}{2}$-transitive linear groups
  Liebeck, Praeger and Saxl, 2018
- Linear groups transitive on "special" (totally isotropic, nondegenerate...) subspaces
  Giudici, Glasby and Praeger, 2023

Remark:

G is a transitive linear group $\iff$ G has 2 orbits: $\{0\}, V \setminus \{0\}$

What if G has 3 orbits on V?

Assume that G has 3 orbits on V. Then either

- G is **irreducible** (**Liebeck, 1987**); or

Linear groups with exactly 3 orbits

Assume that G has 3 orbits on V. Then either

- G is **irreducible** (**Liebeck, 1987**); or
- there is a subspace $W < V$ such that the orbits are

$$\{0\},\ W \setminus \{0\} \text{ and } V \setminus W.$$

Linear groups with exactly 3 orbits

Assume that G has 3 orbits on V. Then either

- G is **irreducible** (**Liebeck, 1987**); or
- there is a subspace $W < V$ such that the orbits are

$$\{0\},\ W \setminus \{0\} \text{ and } V \setminus W.$$

Parabolic subgroup

The maximal subgroup of $\mathrm{GL}(V)$ stabilizing a d-dimensional subspace (denoted by $\mathcal{P}_d(V)$) has exactly 3 orbits on V, and

$$\mathcal{P}_d(\mathbb{F}_p^n) \cong \mathbb{F}_p^{d(n-d)}{:}\big(\mathrm{GL}_d(p) \times \mathrm{GL}_{n-d}(p)\big).$$

Parabolic subgroup

Let $X = \mathcal{P}_d(\mathbb{F}_p^n)$. Then elements in X have matrix form

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & CB^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Let $X = \mathcal{P}_d(\mathbb{F}_p^n)$. Then elements in X have matrix form

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & CB^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Hence $X = Q{:}L \cong \mathbb{F}_p^{d(n-d)}{:}\big(\mathrm{GL}_d(p) \times \mathrm{GL}_{n-d}(p)\big)$, where

- $Q = \left\{ \begin{pmatrix} I & C \\ & I \end{pmatrix} \big| C \in \mathrm{M}_{d(n-d)}(\mathbb{F}_p) \right\}$
- $L = \left\{ \begin{pmatrix} A & \\ & B \end{pmatrix} \big| A \in \mathrm{GL}_d(p), B \in \mathrm{GL}_{n-d}(p) \right\}$

Orbits of parabolic subgroup

Let $X = \mathcal{P}_d(V) = Q{:}L$ stabilizing W.

Orbits of parabolic subgroup

Let $X = \mathcal{P}_d(V) = Q{:}L$ stabilizing W.

- $X^W \cong \mathrm{GL}(W) \implies \{0\}$ and $W \setminus \{0\}$ are 2 orbits of X.

Orbits of parabolic subgroup

Let $X = \mathcal{P}_d(V) = Q{:}L$ stabilizing W.

- $X^W \cong \mathrm{GL}(W) \implies \{0\}$ and $W \setminus \{0\}$ are 2 orbits of X.
- $X^{V/W} \cong \mathrm{GL}(V/W) \implies X$ is transitive on $V/W \setminus \{\bar{0}\}$.

Orbits of parabolic subgroup

Let $X = \mathcal{P}_d(V) = Q{:}L$ stabilizing W.

- $X^W \cong \mathrm{GL}(W) \implies \{0\}$ and $W \setminus \{0\}$ are 2 orbits of X.
- $X^{V/W} \cong \mathrm{GL}(V/W) \implies X$ is transitive on $V/W \setminus \{\bar{0}\}$.
- For any nonzero $\bar{v} \in V/W$, $Q \leqslant X_{\bar{v}}$ is transitive on $\bar{v} = \{w + v : w \in W\}$.

Orbits of parabolic subgroup

Let $X = \mathcal{P}_d(V) = Q{:}L$ stabilizing W.

- $X^W \cong \mathrm{GL}(W) \implies \{0\}$ and $W \setminus \{0\}$ are 2 orbits of X.
- $X^{V/W} \cong \mathrm{GL}(V/W) \implies X$ is transitive on $V/W \setminus \{\bar{0}\}$.
- For any nonzero $\bar{v} \in V/W$, $Q \leqslant X_{\bar{v}}$ is transitive on $\bar{v} = \{w + v : w \in W\}$.

Therefore, X has 3 orbits: $\{0\}, W \setminus \{0\}$ and $V \setminus W$.

Let $X = \mathcal{P}_d(V) = Q{:}L$ stabilizing W.

- $X^W \cong \mathrm{GL}(W) \implies \{0\}$ and $W \setminus \{0\}$ are 2 orbits of X.
- $X^{V/W} \cong \mathrm{GL}(V/W) \implies X$ is transitive on $V/W \setminus \{\bar{0}\}$.
- For any nonzero $\overline{v} \in V/W$, $Q \leqslant X_{\overline{v}}$ is transitive on $\overline{v} = \{w + v : w \in W\}$.

Therefore, X has 3 orbits: $\{0\}, W \setminus \{0\}$ and $V \setminus W$.

Lemma

Subgroup $G \leqslant X$ has exactly 3 orbits if and only if

- *G^W and $G^{V/W}$ are transitive on nonzero vectors of W and V/W, respectively.*
- *$G_{\overline{v}}$ acts on $\overline{v}$ transitively.*

$G \cap Q = 1$

Lemma

Subgroup $G \leqslant X$ has exactly 3 orbits if and only if

- G^W *and* $G^{V/W}$ *are* ***transitive*** *on nonzero vectors of* W *and* V/W*, respectively.*
- $G_{\bar{v}}$ *acts on* $\bar{v}$ *transitively.*

Remarks:

- transitive linear groups are known (**Huppert and Hering**);

$G \cap Q = 1$

Lemma

Subgroup $G \leqslant X$ has exactly 3 orbits if and only if

- *G^W and $G^{V/W}$ are **transitive** on nonzero vectors of W and V/W, respectively.*
- *$G_{\overline{v}}$ acts on $\overline{v}$ transitively.*

Remarks:

- transitive linear groups are known (**Huppert and Hering**);
- if $Q \leqslant G$, then $G_{\overline{v}}$ is transitive on $\overline{v}$.

$G \cap Q = 1$

Lemma

Subgroup $G \leqslant X$ has exactly 3 orbits if and only if

- G^W *and* $G^{V/W}$ *are* ***transitive*** *on nonzero vectors of* W *and* V/W*, respectively.*
- $G_{\overline{v}}$ *acts on* $\overline{v}$ *transitively.*

Remarks:

- transitive linear groups are known (**Huppert and Hering**);
- if $Q \leqslant G$, then $G_{\overline{v}}$ is transitive on $\overline{v}$.

Problem

Classifying $G \leqslant X$ with exactly 3 orbits and $G \cap Q = 1$.

Orbits of Levi subgroups

Recall $L = \left\{ \begin{pmatrix} A & \\ & B \end{pmatrix} \Big| A \in \mathrm{GL}_d(p), B \in \mathrm{GL}_{n-d}(p) \right\}.$

Orbits of Levi subgroups

Recall $L = \left\{ \begin{pmatrix} A & \\ & B \end{pmatrix} \middle| A \in \mathrm{GL}_d(p), B \in \mathrm{GL}_{n-d}(p) \right\}$.

- L stabilizes a direct sum $V = W \oplus U$; and
- $L \cap Q = 1$ and L has 4 orbits:

  $\{0\},\ W\backslash\{0\},\ U\backslash\{0\}$ and $\{w+u \mid 0 \neq w \in W$ and $0 \neq u \in U\}$.

Orbits of Levi subgroups

Recall $L = \left\{ \begin{pmatrix} A & \\ & B \end{pmatrix} \middle| A \in \mathrm{GL}_d(p), B \in \mathrm{GL}_{n-d}(p) \right\}$.

- L stabilizes a direct sum $V = W \oplus U$; and
- $L \cap Q = 1$ and L has 4 orbits:

$\{0\},\ W \backslash \{0\},\ U \backslash \{0\}$ and $\{w+u \mid 0 \neq w \in W$ and $0 \neq u \in U\}$.

Observations

Assume that $G \cap Q = 1$.

- If G is conjugate to a subgroup of L in $X = Q{:}L$, then G has more than 3 orbits.

Orbits of Levi subgroups

Recall $L = \left\{ \begin{pmatrix} A & \\ & B \end{pmatrix} \middle| A \in \mathrm{GL}_d(p), B \in \mathrm{GL}_{n-d}(p) \right\}$.

- L stabilizes a direct sum $V = W \oplus U$; and
- $L \cap Q = 1$ and L has 4 orbits:

$\{0\}$, $W \setminus \{0\}$, $U \setminus \{0\}$ and $\{w + u \mid 0 \neq w \in W$ and $0 \neq u \in U\}$.

Observations

Assume that $G \cap Q = 1$.

- If G is conjugate to a subgroup of L in $X = Q{:}L$, then G has more than 3 orbits.
- If there is only one conjugacy class of complement of Q in $Q{:}G$, then G has more than 3 orbits.

Example: $G \cong \mathrm{GL}_3(2)$

Let $V = \mathbb{F}_2^4$. Then $\mathcal{P}_1(V) \cong \mathbb{F}_2^3{:}\mathrm{GL}_3(2) \cong \mathrm{AGL}_3(2)$.

Then there exist subgroups G_1 and G_2 of $\mathcal{P}_1(V)$ such that

- $G_1 \cong G_2 \cong \mathrm{GL}_3(2)$;

Example: $G \cong \mathrm{GL}_3(2)$

Let $V = \mathbb{F}_2^4$. Then $\mathcal{P}_1(V) \cong \mathbb{F}_2^3{:}\mathrm{GL}_3(2) \cong \mathrm{AGL}_3(2)$.

Then there exist subgroups G_1 and G_2 of $\mathcal{P}_1(V)$ such that

- $G_1 \cong G_2 \cong \mathrm{GL}_3(2)$;
- G_1 has 4 orbits on V; and G_2 has 3 orbits on V.

Example: $G \cong \mathrm{GL}_3(2)$

Let $V = \mathbb{F}_2^4$. Then $\mathcal{P}_1(V) \cong \mathbb{F}_2^3{:}\mathrm{GL}_3(2) \cong \mathrm{AGL}_3(2)$.

Then there exist subgroups G_1 and G_2 of $\mathcal{P}_1(V)$ such that

- $G_1 \cong G_2 \cong \mathrm{GL}_3(2)$;
- G_1 has 4 orbits on V; and G_2 has 3 orbits on V.

$$G_1 = \left\langle \begin{pmatrix} 1 & \\ & A \end{pmatrix}, \begin{pmatrix} 1 & \\ & B \end{pmatrix} \right\rangle \text{ and } G_2 = \left\langle \begin{pmatrix} 1 & e \\ & A \end{pmatrix}, \begin{pmatrix} 1 & \\ & B \end{pmatrix} \right\rangle,$$

where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, and $e = (1, 0, 0)$.

Cohomology group

Let $T \leqslant \mathrm{GL}(V)$. To determine complements of V in $V{:}T$, we use **cohomology group**.

Let $T \leqslant \mathrm{GL}(V)$. To determine complements of V in $V{:}T$, we use **cohomology group**.

Proposition

- $\mathrm{H}^0(T, V) = \mathrm{Fix}(T, V)$.
- $|\mathrm{H}^1(T, V)| = \#\{$conjugacy classes of complements of $V\}$.

Let $T \leqslant \mathrm{GL}(V)$. To determine complements of V in $V{:}T$, we use **cohomology group**.

Proposition

- $\mathrm{H}^0(T, V) = \mathrm{Fix}(T, V)$.
- $|\mathrm{H}^1(T, V)| = \#\{\text{conjugacy classes of complements of } V\}$.

Recall that $G \leqslant \mathcal{P}_d(V) = Q{:}L$. Thus, we have that

if $G \cap Q = 1$ and $|\mathrm{H}^1(G, Q)| = 1$, then G has more than 3 orbits.

Action of L on Q

Recall that $\mathcal{P}_d(V) = Q{:}L$, where

- $Q \cong \mathbb{F}_p^{d\times(n-d)} \cong \mathbb{F}_p^d \otimes \mathbb{F}_p^{n-d}$;
- $L = L_1 \times L_2 \cong \mathrm{GL}_d(p) \times \mathrm{GL}_{n-d}(p)$.

Action of L on Q

Recall that $\mathcal{P}_d(V) = Q{:}L$, where

- $Q \cong \mathbb{F}_p^{d\times(n-d)} \cong \mathbb{F}_p^d \otimes \mathbb{F}_p^{n-d}$;
- $L = L_1 \times L_2 \cong \mathrm{GL}_d(p) \times \mathrm{GL}_{n-d}(p)$.

Lemma

There is a natural $\mathbb{F}_p L$-module structure of Q such that $Q \cong W \otimes V/W$ with

$$(w \otimes \overline{v})^{\ell_1\ell_2} = w^{\ell_1} \otimes \overline{v}^{\ell_2} \text{ for } \ell_i \in L_i \text{ with } i = 1, 2$$

Analyzation: direct product

Suppose that $G = H_1 \times H_2 \cong \mathrm{SL}_5(5) \times \mathrm{SL}_3(5)$ with $G \cap Q = 1$.

- $H_1 := \left\{ \begin{pmatrix} A & * \\ & I \end{pmatrix} : A \in \mathrm{SL}_5(5) \right\}$, $H_2 := \left\{ \begin{pmatrix} I & * \\ & B \end{pmatrix} : B \in \mathrm{SL}_3(5) \right\}$

Suppose that $G = H_1 \times H_2 \cong \mathrm{SL}_5(5) \times \mathrm{SL}_3(5)$ with $G \cap Q = 1$.

- $H_1 := \left\{ \begin{pmatrix} A & * \\ & I \end{pmatrix} : A \in \mathrm{SL}_5(5) \right\}$, $H_2 := \left\{ \begin{pmatrix} I & * \\ & B \end{pmatrix} : B \in \mathrm{SL}_3(5) \right\}$

Lemma (Kunneth formula)

$\mathrm{H}^1(A \times B, W \otimes U) \cong \left(\mathrm{H}^1(A, W) \otimes \mathrm{H}^0(B, U)\right) \oplus \left(\mathrm{H}^0(A, W) \otimes \mathrm{H}^1(B, U)\right)$.

Hence, we have

$$|\mathrm{H}^1(G, Q)| = |\mathrm{H}^1(H_1 \times H_2, W \otimes V/W)| = 1.$$

Analyzation: direct product

Suppose that $G = H_1 \times H_2 \cong \mathrm{SL}_5(5) \times \mathrm{SL}_3(5)$ with $G \cap Q = 1$.

- $H_1 := \left\{ \begin{pmatrix} A & * \\ & I \end{pmatrix} : A \in \mathrm{SL}_5(5) \right\}$, $H_2 := \left\{ \begin{pmatrix} I & * \\ & B \end{pmatrix} : B \in \mathrm{SL}_3(5) \right\}$

Lemma (Kunneth formula)

$\mathrm{H}^1(A \times B, W \otimes U) \cong \left(\mathrm{H}^1(A, W) \otimes \mathrm{H}^0(B, U)\right) \oplus \left(\mathrm{H}^0(A, W) \otimes \mathrm{H}^1(B, U)\right)$.

Hence, we have

$$|\mathrm{H}^1(G, Q)| = |\mathrm{H}^1(H_1 \times H_2, W \otimes V/W)| = 1.$$

In particular, G stabilizes a direct sum $V = W \oplus U \cong \mathbb{F}_5^5 \oplus \mathbb{F}_3^5$.

Actions on W and V/W

Note that

- $G_{(W)}$ is the kernel of G acting on W;
- $G_{(V/W)}$ is the kernel of G acting on V/W;
- $G \cap Q = G_{(W)} \cap G_{(V/W)}$.

Actions on W and V/W

Note that

- $G_{(W)}$ is the kernel of G acting on W;
- $G_{(V/W)}$ is the kernel of G acting on V/W;
- $G \cap Q = G_{(W)} \cap G_{(V/W)}$.

Lemma

If $G \cap Q = 1$ and G has exactly 3 orbits, then at least one of $G_{(W)}$ and $G_{(V/W)}$ is trivial.

Hint: $|\mathrm{H}^1(G_{(W)} \times G_{(V/W)}, Q)| = 1$.

Actions on W and V/W

Lemma

If $G \cap Q = 1$ and G has exactly 3 orbits, then at least one of $G_{(W)}$ and $G_{(V/W)}$ is trivial.

- $G_{(W)} = 1$, $G_{(V/W)} \neq 1$;
- $G_{(W)} = G_{(V/W)} = 1$;
- $G_{(W)} \neq 1$, $G_{(V/W)} = 1$.

Actions on W and V/W

Lemma

If $G \cap Q = 1$ and G has exactly 3 orbits, then at least one of $G_{(W)}$ and $G_{(V/W)}$ is trivial.

- $G_{(W)} = 1,\ G_{(V/W)} \neq 1$; One group appears
- $G_{(W)} = G_{(V/W)} = 1$; No group appears
- $G_{(W)} \neq 1,\ G_{(V/W)} = 1$. One group appears

Theorem (Li, Zhu and Y, 2024+)

If G is a reducible linear group with exactly 3 orbits and $O_p(G) = 1$, then $V \cong \mathbb{F}_2^4$ and $G \cong \mathrm{GL}_3(2)$ is conjugate to G_1 or G_2, where

$$G_1 = \left\langle \left(\begin{array}{c|ccc} 1 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right), \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \right\rangle,$$

$$G_2 = \left\langle \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right), \left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \right\rangle.$$

Case: $G \cap Q \neq 1$

For $V = \mathbb{F}_p^{2n}$, define

$$P := \left\langle \begin{pmatrix} I & k \cdot I \\ & I \end{pmatrix} : k \in \mathbb{F}_p \right\rangle, \; H := \left\langle \begin{pmatrix} A & \\ & A \end{pmatrix} : A \in \mathrm{GL}_n(p) \right\rangle .$$

Set $G := \langle P, H \rangle = P \times H$. We have $G^W \cong G^{V/W} \cong \mathrm{GL}_n(p)$.

For $V = \mathbb{F}_p^{2n}$, define

$$P := \left\langle \begin{pmatrix} I & k \cdot I \\ & I \end{pmatrix} : k \in \mathbb{F}_p \right\rangle, \ H := \left\langle \begin{pmatrix} A & \\ & A \end{pmatrix} : A \in \mathrm{GL}_n(p) \right\rangle.$$

Set $G := \langle P, H \rangle = P \times H$. We have $G^W \cong G^{V/W} \cong \mathrm{GL}_n(p)$.

Let $w = e_1$ and $v = e_{n+1}$. Then $G_{\overline{v}} = P \times H_{\overline{v}}$.

Case: $G \cap Q \neq 1$

For $V = \mathbb{F}_p^{2n}$, define

$$P := \left\langle \begin{pmatrix} I & k \cdot I \\ & I \end{pmatrix} : k \in \mathbb{F}_p \right\rangle, \; H := \left\langle \begin{pmatrix} A & \\ & A \end{pmatrix} : A \in \mathrm{GL}_n(p) \right\rangle.$$

Set $G := \langle P, H \rangle = P \times H$. We have $G^W \cong G^{V/W} \cong \mathrm{GL}_n(p)$.

Let $w = e_1$ and $v = e_{n+1}$. Then $G_{\overline{v}} = P \times H_{\overline{v}}$.

- $H_{\overline{v}}$ fixes w and v.
- P stabilizes $\{kw + v : k \in \mathbb{F}_p\}$.

Case: $G \cap Q \neq 1$

For $V = \mathbb{F}_p^{2n}$, define

$$P := \left\langle \begin{pmatrix} I & k \cdot I \\ & I \end{pmatrix} : k \in \mathbb{F}_p \right\rangle, \ H := \left\langle \begin{pmatrix} A & \\ & A \end{pmatrix} : A \in \mathrm{GL}_n(p) \right\rangle.$$

Set $G := \langle P, H \rangle = P \times H$. We have $G^W \cong G^{V/W} \cong \mathrm{GL}_n(p)$.

Let $w = e_1$ and $v = e_{n+1}$. Then $G_{\overline{v}} = P \times H_{\overline{v}}$.

- $H_{\overline{v}}$ fixes w and v.
- P stabilizes $\{kw + v : k \in \mathbb{F}_p\}$.

$\implies$ $G_{\overline{v}}$ stabilizes $\{kw + v : k \in \mathbb{F}_p\} \subsetneq \overline{v}$

$\implies$ G has more than 3 orbits.

For $V = \mathbb{F}_p^{2n}$, set $G = \langle P, H \rangle = P{:}H$, where

- $P := \left\langle \begin{pmatrix} I & M \\ & I \end{pmatrix} : M^T = -M \text{ for } M \in \mathrm{M}_{n\times n}(\mathbb{F}_p) \right\rangle$, and
- $H := \left\langle \begin{pmatrix} A & \\ & (A^T)^{-1} \end{pmatrix} : A \in \mathrm{GL}_n(p) \right\rangle$.

Then G has more than 3 orbits and $G \cap Q = P \cong \mathbb{F}_p^{n(n-1)/2}$.

Theorem

Assume that $G \leqslant \mathcal{P}_d(V)$ is non-solvable such that both G^W and $G^{V/W}$ are transitive linear groups and $O_p(G) \neq 1$.

Theorem

Assume that $G \leqslant \mathcal{P}_d(V)$ *is non-solvable such that both* G^W *and* $G^{V/W}$ *are transitive linear groups and* $O_p(G) \neq 1$.

If G *has more than* 3 *orbits, then*

- $\dim V = 2 \dim W$ *and* $H^{V/W} \cong H^W$, *where* $H = G^{(\infty)}$.

Theorem

Assume that $G \leqslant \mathcal{P}_d(V)$ *is non-solvable such that both* G^W *and* $G^{V/W}$ *are transitive linear groups and* $O_p(G) \neq 1$.

If G *has more than* 3 *orbits, then*

- $\dim V = 2\dim W$ *and* $H^{V/W} \cong H^W$, *where* $H = G^{(\infty)}$.
- *there exists a basis of* V *and* $\varphi \in \mathrm{Out}(H^W)$ *such that each* $h \in H$ *has matrix form*

$$\mathcal{M}(h) = \begin{pmatrix} A & * \\ 0 & A^{\varphi} \end{pmatrix},$$

Theorem

Assume that $G \leqslant \mathcal{P}_d(V)$ is non-solvable such that both G^W and $G^{V/W}$ are transitive linear groups and $O_p(G) \neq 1$.

If G has more than 3 orbits, then

- $\dim V = 2\dim W$ *and* $H^{V/W} \cong H^W$, *where* $H = G^{(\infty)}$.
- *there exists a basis of V and $\varphi \in \mathrm{Out}(H^W)$ such that each $h \in H$ has matrix form*

$$\mathcal{M}(h) = \begin{pmatrix} A & * \\ 0 & A^{\varphi} \end{pmatrix},$$

  where $\varphi = 1$; or

  - $H^W \cong \mathrm{SL}_m(p^f)$ $(m \geqslant 3)$ *or* A_7 ($\dim V = 8$ *and* $p = 2$), *and φ is the* ***transpose inverse****; or*
  - $H^W \cong \mathrm{Sp}_4(2^f)$ *with $\varphi = \gamma$* ***(graph automorphism)***.

Rank 3 groups

The **rank** of transitive group $X \leqslant \mathrm{Sym}(\Omega)$ is the number of orbits of X_α for $\alpha \in \Omega$.

Rank 3 groups

The **rank** of transitive group $X \leqslant \mathrm{Sym}(\Omega)$ is the number of orbits of X_α for $\alpha \in \Omega$.

Example

If $G \leqslant \mathrm{GL}(V)$ has exactly 3 orbits on V, then $\overline{V}{:}G \leqslant \mathrm{AGL}(V)$ is a rank 3 permutation group on V.

Rank 3 groups

The **rank** of transitive group $X \leqslant \mathrm{Sym}(\Omega)$ is the number of orbits of X_α for $\alpha \in \Omega$.

Example

If $G \leqslant \mathrm{GL}(V)$ has exactly 3 orbits on V, then $\overline{V}{:}G \leqslant \mathrm{AGL}(V)$ is a rank 3 permutation group on V.

- Primitive rank 3 groups were classified by Bannai, Foulser, Kantor, Liebeck, Liebler, and Saxl (1969 - 1986).

Rank 3 groups

The **rank** of transitive group $X \leqslant \mathrm{Sym}(\Omega)$ is the number of orbits of X_α for $\alpha \in \Omega$.

Example

If $G \leqslant \mathrm{GL}(V)$ has exactly 3 orbits on V, then $\overline{V}{:}G \leqslant \mathrm{AGL}(V)$ is a rank 3 permutation group on V.

- Primitive rank 3 groups were classified by Bannai, Foulser, Kantor, Liebeck, Liebler, and Saxl (1969 - 1986).
- Many researches on imprimitive rank 3:
  - **Quasiprimitive**: Devillers, Giudici, Li, Pearce, and Praeger (2011)
  - **Innately transitive**: Baykalov, Devillers, and Praeger (2023)
  - **"Some" semiprimitive**: Huang, Li, and Zhu (2024+)

Let G be an imprimitive rank 3 group.

Lemma

- *G has a unique non-trivial block system $\mathcal{B}$; and*
- *both $G^{\mathcal{B}}$ and G_B^B are 2-transitive groups.*

Let G be an imprimitive rank 3 group.

Lemma

- *G has a unique non-trivial block system $\mathcal{B}$; and*
- *both $G^{\mathcal{B}}$ and G_B^B are 2-transitive groups.*

- **G_B^B is almost simple**

  Devillers, Giudici, Li, Pearce and Praeger (2011).

Imprimitive rank 3 groups

Let G be an imprimitive rank 3 group.

Lemma

- *G has a unique non-trivial block system $\mathcal{B}$; and*
- *both $G^{\mathcal{B}}$ and G_B^B are 2-transitive groups.*

- **G_B^B is almost simple**

  Devillers, Giudici, Li, Pearce and Praeger (2011).

- **G_B^B is affine**

  Huang, Li and Zhu (2024+) divide them into 4 sections.

G_B^B **is affine**: divided into 4 cases (Huang, Li and Zhu, 2024+)

G_B^B **is affine**: divided into 4 cases (Huang, Li and Zhu, 2024+)
A particular case:

$N \lhd G \leqslant N{:}\mathrm{Aut}(N)$, where $\mathrm{Aut}(N)$ has most 3 orbits on N.

- $\mathrm{Aut}(N)$ has 3 orbits on N, and $(N, \mathrm{Aut}(N))$ is classified by Li and Zhu (2024+);
- $N \cong \mathbb{Z}_p^n$ and $G_\alpha \leqslant \mathrm{GL}_n(p)$ is a reducible group with exactly 3 orbits on N.