Connections of the classical Erdős-Ko-Rado theorem with the theory of combinatorial designs

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Outline

- Maximal cliques in graphs
- Classical Erdős–Ko–Rado theorem and its formulation in terms of cocliques in Kneser graphs
- Replacement of Kneser graphs by the block graphs of 2-designs
- Replacement of Kneser graphs by the block graph of orthogonal arrays

Introducing myself

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Maximal cliques in simple graphs

We consider simple graphs, that is graphs, without loops and multiple edges.

A subset C of vertices in a graph Γ is called a clique if any two vertices in C are adjacent.

Problem 1 (A general problem in graph theory)

For a graph Γ, determine all its cliques.

A clique in a graph is called maximal if it is not included in another clique of larger size.

To solve Problem 1, it suffices to solve the following problem.

Problem 2 (A general problem in graph theory)

For a graph Γ, determine all its maximal cliques.

Cliques and cocliques as complementary objects

For a graph Γ, the complement of Γ, denoted by $\overline{\Gamma}$, is the graph with the same vertex set as Γ and inverted adjacency relation.

A subset C of vertices in a graph Γ is called a coclique (independent set), if any two vertices in C are not adjacent.

The cliques in Γ are in one-to-one correspondence with cocliques in $\overline{\Gamma}$; in particular, maximal cliques in Γ are in one-to-one correspondence with maximal cocliques in $\overline{\Gamma}$.

A clique C of size s in a graph Γ is called maximum if Γ has no cliques of size larger than s.

Every maximum clique is always maximal, but not every maximal clique is maximum. Maximum cliques are just largest (with respect to the size) maximal cliques.

Erdős–Ko–Rado theorem

The Erdős–Ko–Rado theorem, one of the fundamental results in combinatorics, provides information about systems of intersecting sets. A family $\mathcal{A}$ of subsets of a ground set — it might as well be $\{1, \ldots, n\}$ — is intersecting if any two sets in $\mathcal{A}$ have at least one point in common.

Theorem 1 (Erdős–Ko–Rado, 1961)

Let k and n be integers with $n \geq 2k$. If $\mathcal{A}$ is an intersecting family of k-subsets of $\{1, \ldots, n\}$, then

$$|\mathcal{A}| \leq \binom{n-1}{k-1}.$$

Moreover, if $n > 2k$, equality holds if and only if $\mathcal{A}$ consists of all the k-subsets that contain a given point from $\{1, \ldots, n\}$.

An intersecting family $\mathcal{A}$ consisting of all the k-subsets that contain a given point from $\{1, \ldots, n\}$ is called canonical.

Extensions of Erdős–Ko–Rado theorem

This theorem has two parts: a bound and a characterisation of families that meet the bound, and states that a maximum intersecting family is necessarily canonical.

One reason this theorem is so important is that it has many interesting extensions. To address these, we first translate it to a question in graph theory. The Kneser graph $K(n, k)$ has all k-subsets of $\{1, \dots, n\}$ as its vertices, and two k-subsets are adjacent if they are disjoint. (We assume $n \geq 2k$ to avoid trivialities.)

Then an intersecting family of k-subsets is a coclique in the Kneser graph, and we see that the EKR theorem characterises the cocliques of maximum size in the Kneser graph.

So we can seek to extend the EKR theorem by replacing the Kneser graphs by other interesting families of graphs.

EKR-type results

The classical Erdős-Ko-Rado theorem [EKR61] classified maximum intersecting families of k-element subsets of $\{1, 2, \ldots, n\}$ when $n \geq 2k + 1$.

Since then, EKR-type results refer to understanding maximum intersecting families in a broader context, and more generally, classifying extremal configurations in other domains. The book [GM15] by Godsil and Meagher provides an excellent survey on the modern algebraic approaches to proving EKR-type results for permutations, set systems, orthogonal arrays, and so on.

[EKR61] P. Erdős, C. Ko, and R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser. (2) 12 (1961), 313–320.

[GM15] C. D. Godsil, K. Meagher, *Erdős-Ko-Rado Theorems: Algebraic Approaches*, Cambridge University Press (2015).

Strongly regular graphs

A graph Γ is called k-regular if there exists an integer $k \geq 0$ such that each vertex in Γ has exactly k neighbours.

A graph on v vertices is called a strongly regular graph with parameters (v, k, λ, μ) if:
(a) it is k-regular;
(b) each pair of adjacent vertices in the graph have exactly λ common neighbours;
(c) each pair of distinct nonadjacent vertices in the graph have exactly μ common neighbours.

Delsarte-Hoffman bound

Every non-trivial strongly regular graph has exactly three distinct eigenvalues, which can be expressed in terms of the parameters.

For the clique number $\omega(\Gamma)$ of a strongly regular graph Γ, the Delsarte-Hoffman bound holds:

$$\omega(\Gamma) \leq 1 - \frac{k}{\theta_{\min}},$$

where $\theta_{\min}$ is the smallest eigenvalue of Γ.

A clique in a strongly regular graph whose size lies on the Delsarte-Hoffman bound is called a Delsarte clique.

Projective planes

An projective plane is a system of points and lines that satisfy the following axioms:

- Any two distinct points lie on a unique line.
- Any two distinct lines intersect in a unique point.
- There exist three non-collinear points (points not on a single line).

In this lecture we are interested in finite projective planes (that is, projective planes having finitely many points).

Properties of finite projective planes

If the number of points in an projective plane is finite, then if one line of the plane contains $n + 1$ points for some positive integer n then:

- each line contains $n + 1$ points,
- each point is contained in $n + 1$ lines,
- there are $n^2 + n + 1$ points in all, and
- there is a total of $n^2 + n + 1$ lines.

The number n is called the order of the projective plane.

Projective plane $\mathrm{PG}(2, q)$

Let q be a prime power and V be a 3-dimensional vector space over the finite field $\mathbb{F}_q$.

- The points of $\mathrm{PG}(2, q)$ are the 1-dimensional subspaces of V.
- The lines of $\mathrm{PG}(2, q)$ are the 2-dimensional subspaces of V.
- The incidence between the points and the lines is given by the natural inclusion.

In a similar way, the projective space $\mathrm{PG}(d, q)$ can be defined for any integer $d \geq 2$.

Affine planes

An affine plane is a system of points and lines that satisfy the following axioms:

- Any two distinct points lie on a unique line.
- Given any line and any point not on that line there is a unique line which contains the point and does not meet the given line. (Playfair's axiom)
- There exist three non-collinear points (points not on a single line).

In an affine plane, two lines are called parallel if they are equal or disjoint. Using this definition, Playfair's axiom above can be replaced by:

- Given a point and a line, there is a unique line which contains the point and is parallel to the line.

The familiar Euclidean plane is an affine plane. In this lecture we are interested in finite affine planes (that is, affine planes having finitely many points).

Properties of finite affine planes

If the number of points in an affine plane is finite, then if one line of the plane contains n points then:

- each line contains n points,
- each point is contained in $n + 1$ lines,
- there are n^2 points in all, and
- there is a total of $n^2 + n$ lines.

The number n is called the order of the affine plane.

Affine plane $\mathrm{AG}(2, q)$

Let q be a prime power and W be a 2-dimensional vector space over the finite field $\mathbb{F}_q$.

- The points of $\mathrm{AG}(2, q)$ are all the cosets of the 0-dimensional subspace of W.
- The lines of $\mathrm{AG}(2, q)$ are all the cosets of the 1-dimensional subspaces of W.
- The incidence between the points and the lines is given by the natural inclusion.

In a similar way, the affine space $\mathrm{AG}(d, q)$ can be defined for any integer $d \geq 2$.

The goal of this lecture

In order to extend the Erdős–Ko–Rado theorem to another graphs, we need to define the notion of 'intersecting' for the vertices of the graphs we consider. Once we have done this, we can define the notion of canonical intersecting families in these graphs.

In this lecture we consider two important families of strongly regular graphs for which the notion of canonical intersecting families (canonical cliques) is well defined: the block graphs of 2-$(n, m, 1)$ designs and the block graphs of orthogonal arrays.

2-designs

A 2-$(n, m, 1)$ design is a collection of m-sets of an n-set with the property that every pair from the n-set is in exactly one m-set.

A specific 2-$(n, m, 1)$ design is denoted by $(V, \mathcal{B})$, where V is the n-set (which we call the base set) and $\mathcal{B}$ is the collection of m-sets — these are called the blocks of the design.

A 2-$(n, m, 1)$ design may also be called a 2-design.

A simple counting argument shows that the number of blocks in a 2-$(n, m, 1)$ design is $\frac{n(n-1)}{m(m-1)}$ and each element of V occurs in exactly $\frac{n-1}{m-1}$ blocks (this is usually called the replication number).

Example of a 2-(13,3,1) design (a Steiner triple system)

- The point-set is the cyclic group $\mathbb{Z}/13\mathbb{Z}$
- The triples are the twenty-six 3-sets that are cyclically generated (mod 13) by the base blocks $\{1,3,9\}, \{2,5,6\}$ under the transformation $x \mapsto x+1$.

$$
\begin{array}{cc}
\{1,3,9\} & \{2,5,6\} \\
\{2,4,10\} & \{3,6,7\} \\
\{3,5,11\} & \{4,7,8\} \\
\{4,6,12\} & \{5,8,9\} \\
\{5,7,0\} & \{6,9,10\} \\
\{6,8,1\} & \{7,10,11\} \\
\{7,9,2\} & \{8,11,12\} \\
\{8,10,3\} & \{9,12,0\} \\
\{9,11,4\} & \{10,0,1\} \\
\{10,12,5\} & \{11,1,2\} \\
\{11,0,6\} & \{12,2,3\} \\
\{12,1,7\} & \{0,3,4\} \\
\{0,2,8\} & \{1,4,5\}
\end{array}
$$

EKR-type theorem for 2-designs

The blocks of a 2-design are a set system, and every pair from the base set occurs in exactly one block. Thus two distinct blocks of a 2-design must have intersection size 0 or 1. An intersecting set system from a 2-design is a set of blocks from the design in which any two have intersection of size exactly 1. (Note that a 2-design can be viewed as a combinatorial approximation of the set of k-subsets of an n-element set appearing in the EKR theorem.)

A question that naturally arises: what is the largest possible such set? Clearly if we take the collection of all blocks that contain a fixed element, we will have a system of size $\frac{n-1}{m-1}$.

An EKR-type theorem for 2-designs would state that this is the largest possible set of intersecting blocks and determine the conditions when the only intersecting sets of blocks that has this size is the set of all blocks that contain a fixed element. (The first result would be the bound in the EKR theorem, and the second would be the characterisation.)

Block graph of a 2-$(n, m, 1)$ design

The block graph of a 2-$(n, m, 1)$ design $(V, \mathcal{B})$ is the graph with the blocks of the design as the vertices in which two blocks are adjacent if and only if they intersect.

In a 2-design, any two blocks that intersect meet in exactly one point. The block graph of a design $(V, \mathcal{B})$ is denoted by $X_{(V,\mathcal{B})}$.

Alternatively, we could define a graph on the same vertex set in which two vertices are adjacent if and only if the blocks do not intersect — this graph is simply the complement of the block graph.

A clique in the block graph $X_{(V,\mathcal{B})}$ (or a coclique in its complement) is an intersecting set system from $(V, \mathcal{B})$.

Designs that are not symmetric are non-trivial

Fisher's inequality implies that the number of blocks in a 2-design is at least n; if equality holds, the design is said to be symmetric and the block graph of a symmetric 2-design is the complete graph K_n. To avoid this trivial case, we assume that our designs are not symmetric.

Theorem 2 (Well-known)

The block graph of a 2-$(n, m, 1)$ design (that is not symmetric) is strongly regular with parameters

$$\left(\frac{n(n-1)}{m(m-1)}, \frac{m(n-m)}{m-1}, (m-1)^2 + \frac{n-1}{m-1} - 2, m^2\right).$$

Canonical cliques of the block graph of a non-trivial design

The Delsarte bound says that a clique in the block graph of a 2-$(n, m, 1)$ design has size at most $\frac{n-1}{m-1}$.

It is not difficult to construct a clique of this size: for any $i \in \{1, \ldots, n\}$ let S_i be the collection of all blocks in the design that contain i. We call the cliques S_i the canonical cliques of the block graph.

From this, we know that a set of intersecting blocks in a 2-design is no larger than the set of all blocks that contain a common point — this is the bound for an EKR-type theorem for the blocks in a design.

Example of a 2-(13,3,1) design and canonical cliques

$$\begin{array}{cc}
\{1,3,9\} & \{2,5,6\} \\
\{2,4,10\} & \{3,6,7\} \\
\{3,5,11\} & \{4,7,8\} \\
\{4,6,12\} & \{5,8,9\} \\
\{5,7,0\} & \{6,9,10\} \\
\{6,8,1\} & \{7,10,11\} \\
\{7,9,2\} & \{8,11,12\} \\
\{8,10,3\} & \{9,12,0\} \\
\{9,11,4\} & \{10,0,1\} \\
\{10,12,5\} & \{11,1,2\} \\
\{11,0,6\} & \{12,2,3\} \\
\{12,1,7\} & \{0,3,4\} \\
\{0,2,8\} & \{1,4,5\}
\end{array}$$

For a given point, there are exactly six triples containing it. Each such six triples form a canonical clique. The six triples containing 1:

$$\{1,3,9\}, \{6,8,1\}, \{12,1,7\}, \{10,0,1\}, \{11,1,2\}, \{1,4,5\}.$$

A sufficient condition for block graphs of 2-designs to have only canonical maximum cliques

It is not known for which designs the canonical intersecting sets are the only maximum intersecting sets. Godsil & Meagher offer a partial result.

Theorem 3 ([GM15, Theorem 5.3.4])

If a clique in the block graph of a 2-$(n, m, 1)$ design does not consist of all the blocks that contain a given point, then its size is at most $m^2 - m + 1$.

A corollary of this is an analogue of the EKR theorem, with the characterisation of maximum families, for intersecting sets of blocks in a 2-$(n, m, 1)$ design.

Corollary 1 ([GM15, Corollary 5.3.5])

The only cliques of size $\frac{n-1}{m-1}$ in the block graph $X_{(V,\mathcal{B})}$ of a 2-$(n, m, 1)$ design with $n > m^3 - 2m^2 + 2m$ are the sets of blocks that contain a given point i in $\{1, \ldots, n\}$.

Example of the block graph of a 2-design for which there are non-canonical maximum cliques

The characterisation in this corollary may fail if $n \leq m^3 - 2m^2 + 2m$.

For example, consider the projective geometry $PG(3, 2)$. The points of this geometry can be identified with the 15 nonzero vectors in a 4-dimensional vector space V over $GF(2)$, and the lines with the 35 subspaces of dimension 2. This gives us a design with parameters 2-(15, 3, 1), where each block consists of the three nonzero vectors in a 2-dimensional subspace. The parameters of this design correspond to equality case in the inequality above.

There are exactly 15 subspaces of V with dimension 3, and each such subspace contains exactly seven points and exactly seven lines and so provides a copy of the projective plane of order two.

Each point of the design lies on exactly seven lines, and this provides the family of 15 canonical cliques of size 7.

In addition, the seven lines in any one of the 15 projective planes form a non-canonical maximum clique of size 7.

Equality case

The characterisation in Corollary 1 may fail if $n \leq m^3 - 2m^2 + 2m$.

Theorem 4 ([GM15, Exercise 5.7])

In case $n = m^3 - 2m^2 + 2m$, *a non-canonical clique in the block graph of a 2-*$(n, m, 1)$ *design necessarily forms a* $(m^2 - m + 1, m, 1)$ *subdesign (which is a projective plane of order* $m - 1$*).*

Open problems formulated in the book

Problem 3 ([GM15, Problem 16.3.1])

Determine a characterisation of the 2-$(n, m, 1)$ designs, based only on the parameters of the design, for which the only maximum cliques in the block graph are the canonical cliques.

We have considered an example of a design with a block graph that has maximum cliques which are not canonical cliques. These maximum cliques have an interesting structure — namely, they form a subdesign isomorphic to the Fano plane. It is not clear if this a result of a wider phenomenon.

Problem 4 ([GM15, Problem 16.3.2])

When the block graph of a design has maximum cliques that are not canonical, are the non-canonical cliques isomorphic to smaller designs?

Problem 5

Determine all the maximum cliques in the block graph for any 2-$(n, m, 1)$ design.

Further investigation

The stated problems suggest a comprehensive investigation of 2-$(n, m, 1)$ designs such that n and m satisfy the inequality $n < m^3 - 2m^2 + 2m$.

Very recently, we obtained [GK23] a negative answer to Problem 4 by giving an explicit counterexample.

[GK23] S. Goryainov, E. V. Konstantinova, *Non-canonical maximum cliques without a design structure in the block graphs of 2-designs*, November 2023.
`https://arxiv.org/abs/2311.01190`

A 2-$(66, 6, 1)$ design (I)

Consider the 2-$(66, 6, 1)$ design constructed in [D80]. The construction of this design can also be found in [CD07, p. 76].

The point set is $\mathcal{P} = \mathbb{Z}_{13} \times (\mathbb{Z}_3 \cup \{a, b\}) \cup \{\infty\}$.

[D80] R. H. F. Denniston, *A Steiner system with a maximal arc*, Ars Combin. 9 (1980) 247–248.

[CD07] C. J. Colbourn, Jeffrey H. Dinitz (editors), *Handbook of Combinatorial Designs*, Chapman & Hall/CRC, Boca Raton, FL, second edition, 2007.

A 2-$(66, 6, 1)$ design (II)

To define the block set, consider the following eleven basic blocks:

$$\begin{aligned}
B_1 &= \{2_0, 5_0, 4_1, 9_1, 0_a, 6_a\}, \\
B_2 &= \{1_0, 2_0, 6_0, 12_2, 5_b, 8_b\}, \\
B_3 &= \{6_1, 2_1, 12_2, 1_2, 0_a, 5_a\}, \\
B_4 &= \{3_1, 6_1, 5_1, 10_0, 2_b, 11_b\}, \\
B_5 &= \{5_2, 6_2, 10_0, 3_0, 0_a, 2_a\}, \\
B_6 &= \{9_2, 5_2, 2_2, 4_1, 6_b, 7_b\}, \\
B_7 &= \{7_0, 9_0, 10_1, 1_2, 3_a, 4_b\}, \\
B_8 &= \{2_a, 6_a, 5_a, 4_b, 12_b, 10_b\}, \\
B_9 &= \{8_1, 1_1, 4_2, 3_0, 9_a, 12_b\}, \\
B_{10} &= \{11_2, 3_2, 12_0, 9_1, 1_a, 10_b\}, \\
B_{11} &= \{\infty, 0_0, 0_1, 0_2, 0_a, 0_b\}.
\end{aligned}$$

A 2-$(66, 6, 1)$ design (III)

The 143 blocks of the block set $\mathcal{B}$ of the design are then obtained by developing modulo 13 the $\mathbb{Z}_{13}$-components of all points in the basic blocks. More precisely, for any $e \in \mathbb{Z}_{13}$ and basic block B_i, denote by B_i^e the set of points obtained from the points of B_i by adding e modulo 13 to the $\mathbb{Z}_{13}$-component of each non-infinity point of B_i. Then, for the block set $\mathcal{B}$ of the design, we have:

$$\mathcal{B} = \bigcup_{i=1}^{11} \bigcup_{e \in \mathbb{Z}_{13}} B_i^e.$$

Non-canonical cliques without a design structure (I)

Consider the following blocks:

$$
\begin{aligned}
B_1^{11} &= \{0_0, 3_0, 2_1, 7_1, 11_a, 4_a\},\\
B_2^{1} &= \{2_0, 3_0, 7_0, 0_2, 6_b, 9_b\},\\
B_3^{1} &= \{7_1, 3_1, 0_2, 2_2, 1_a, 6_a\},\\
B_4^{10} &= \{0_1, 3_1, 2_1, 7_0, 12_b, 8_b\},\\
B_5^{10} &= \{2_2, 3_2, 7_0, 0_0, 10_a, 12_a\},\\
B_6^{11} &= \{7_2, 3_2, 0_2, 2_1, 4_b, 5_b\},\\
B_7^{6} &= \{0_0, 2_0, 3_1, 7_2, 9_a, 10_b\},\\
B_9^{12} &= \{7_1, 0_1, 3_2, 2_0, 8_a, 11_b\},\\
B_{10}^{4} &= \{2_2, 7_2, 3_0, 0_1, 5_a, 1_b\},\\
B_{11}^{0} &= \{\infty, 0_0, 0_1, 0_2, 0_a, 0_b\},\\
B_{11}^{2} &= \{\infty, 2_0, 2_1, 2_2, 2_a, 2_b\},\\
B_{11}^{3} &= \{\infty, 3_0, 3_1, 3_2, 3_a, 3_b\},\\
B_{11}^{7} &= \{\infty, 7_0, 7_1, 7_2, 7_a, 7_b\}.
\end{aligned}
$$

Non-canonical cliques without a design structure (II)

Put

$$C_1 = \{B_1^{11}, B_2^1, B_3^1, B_4^{10}, B_5^{10}, B_6^{11}, B_7^6, B_9^{12}, B_{10}^4, B_{11}^0, B_{11}^2, B_{11}^3, B_{11}^7\}.$$

Proposition 1 (G., Konstantinova, 2023)

The set C_1 is a non-canonical maximum clique in the block graph of the design $(\mathcal{P}, \mathcal{B})$.

For any $e \in \mathbb{Z}_{13}$, put

$$C_1^e = \{B^e \mid B \in C_1\}.$$

Corollary 1 (G., Konstantinova, 2023)

For any $e \in \mathbb{Z}_{13}$, the set C_1^e forms a non-canonical maximum clique in the block graph of the design $(\mathcal{P}, \mathcal{B})$.

Thus, Corollary 1 gives thirteen non-canonical maximum cliques.

Non-canonical cliques without a design structure (III)

Consider the set of blocks

$$C_2 = \bigcup_{e \in \mathbb{Z}_{13}} \{B_8^e\},$$

obtained by developing modulo 13 the $\mathbb{Z}_{13}$-components of all points in the basic block B_8.

Proposition 2 (G., Konstantinova, 2023)

The set C_2 is a non-canonical maximum clique in the block graph of the design $(\mathcal{P}, \mathcal{B})$.

Non-canonical cliques without a design structure (IV)

The blocks in C_2 are:

$$
\begin{aligned}
B_8^0 &= \{2_a, 6_a, 5_a, 4_b, 12_b, 10_b\},\\
B_8^1 &= \{3_a, 7_a, 6_a, 5_b, 0_b, 11_b\},\\
B_8^2 &= \{4_a, 8_a, 7_a, 6_b, 1_b, 12_b\},\\
B_8^3 &= \{5_a, 9_a, 8_a, 7_b, 2_b, 0_b\},\\
B_8^4 &= \{6_a, 10_a, 9_a, 8_b, 3_b, 1_b\},\\
B_8^5 &= \{7_a, 11_a, 10_a, 9_b, 4_b, 2_b\},\\
B_8^6 &= \{8_a, 12_a, 11_a, 10_b, 5_b, 3_b\},\\
B_8^7 &= \{9_a, 0_a, 12_a, 11_b, 6_b, 4_b\},\\
B_8^8 &= \{10_a, 1_a, 0_a, 12_b, 7_b, 5_b\},\\
B_8^9 &= \{11_a, 2_a, 1_a, 0_b, 8_b, 6_b\},\\
B_8^{10} &= \{12_a, 3_a, 2_a, 1_b, 9_b, 7_b\},\\
B_8^{11} &= \{0_a, 4_a, 3_a, 2_b, 10_b, 8_b\},\\
B_8^{12} &= \{1_a, 5_a, 4_a, 3_b, 11_b, 9_b\}.
\end{aligned}
$$

Non-canonical cliques without a design structure (V)

We then verified by Magma that the block graph of $(\mathcal{P}, \mathcal{B})$ has only these fourteen non-canonical maximum cliques. Finally, it follows from [CD07, p. 72, Table 3.3] that a 2-(39,6,1) design (the total number of points in the union of the blocks from C_1 is 39) and a 2-(26,6,1) design (the total number of points in the union of the blocks from C_2 is 26) do not exist, which implies that neither of the fourteen non-canonical maximum cliques has a design structure.

A new research direction

Thus, the following theorem holds.

Theorem 2 ([GK23, Theorem 1])

A non-canonical maximum clique in the block graph of a 2-design does not necessarily have a design structure.

In [CD07, p. 72, Table 3.3], the necessary and sufficient conditions for the existence of a 2-$(n, m, 1)$ design with $m \leq 9$ are given. For $m \geq 10$, much less is known. However, as was previously discussed, non-canonical maximum cliques in the block graph of a 2-$(n, m, 1)$ design may exist only if $n \leq m^3 - 2m^2 + 2m$. This means that for a fixed value of m, there exist only finitely many 2-$(n, m, 1)$ designs whose block graphs may have non-canonical maximum cliques. We thus formulate the following open problem.

Problem 6

Does there exist an infinite family of 2-designs whose block graphs have non-canonical maximum cliques without a design structure?

Known infinite families of 2-designs

Let us have a look at the known infinite families of 2-$(n, m, 1)$ designs (equivalently, Steiner systems $S(t, m, n)$ where $t = 2$) with growing m. For this, let us have a look at [CD07, p. 103, item 5.11], which provides the following four known infinite families of Steiner systems $S(2, m, n)$:

1. $S(2, q + 1, q^t + \ldots + q + 1)$, q a prime power, $t \geq 2$ (projective 2-designs);
2. $S(2, q, q^t)$, q a prime power, $t \geq 2$ (affine 2-designs);
3. $S(2, q + 1, q^3 + 1)$, q a prime power (unitals);
4. $S(2, 2^r, 2^{r+s} + 2^r - 2^s)$, $2 \leq r < s$ (Denniston designs).

For family 3, the sufficient condition for n and m is never satisfied (non-canonical maximum cliques might exist), but in 2015 De Boeck showed that only canonical cliques exist in the block graph of any unital.

For families 1 and 2 (geometric designs), all maximum cliques also known and we further discuss their structure.

Projective 2-designs

Projective designs on points and lines in $\mathrm{PG}(d, q)$:

- If $d \geq 4$, the inequality is satisfied, and we have only canonical cliques in the block graph.
- If $d = 3$, we have equality in this inequality (the inequality is not satisfied). The block graph in this case is the Grassmann graph $J_q(4, 2)$. It is well-known that non-canonical maximum cliques exist in this case, and each of them corresponds to the lines in a subplane (that is, all non-canonical cliques have a design structure).
- If $d = 2$, we have a symmetric design that is given by the lines in a projective plane (the block graph is a clique and we have nothing to do).

Affine 2-designs

Affine designs on points and lines in $\mathrm{AG}(d, q)$:

- If $d \geq 3$, the inequality is satisfied, and we have only canonical cliques in the block graph.
- If $d = 2$, the inequality does not hold, but the design we have is given by the lines in an affine plane (the block graph is a complete multipartite graph and all cliques are easy to describe).

Maximum cliques in the block graphs of projective and affine designs

Thus, maximum cliques in the block graphs of projective and affine designs on points and lines are known. In particular, the only non-trivial case when non-canonical cliques exist is the case of the design on the points and lines of $\mathrm{PG}(3, q)$ (in this case each non-canonical clique has a design structure).

Denniston designs (I)

The only infinite family of 2-designs for which maximum cliques are not known in general is the family of Denniston designs: $S(2, 2^r, 2^{r+s} + 2^r - 2^s)$, $2 \leq r < s$.

The sufficient condition (implying that all maximum cliques are canonical) is not satisfied if and only if $s < 2r$ holds.

Construction:

- Let $\mathrm{AG}(2, q)$ be the Desarguesian finite affine plane of order $q = 2^s$.
- Let H be a subgroup of order 2^r in $\mathbb{F}_q^+$.
- Let
$$f(x, y) = ax^2 + hxy + by^2 \in \mathbb{F}_q[x, y]$$
be an irreducible polynomial over $\mathbb{F}_q$.
- Let
$$\Omega = \{(x, y) : f(x, y) \in H\}.$$
- Any line of $\mathrm{AG}(2, 2^s)$ intersects Ω in 0 or 2^r points.
- The point set of the Denniston design is Ω; the blocks of the Denniston design are all the intersection 2^r-sets.

Denniston designs (II)

We have checked the designs from this construction with parameters $(r, s) \in \{(2,3),(3,4),(3,5),(4,5),(4,6),(4,7)\}$ and found that only canonical cliques exist.

Problem 7

Does there exist a Denniston design whose block graph has a non-canonical maximum clique?

Orthogonal arrays and their block graphs

An orthogonal array $OA(m,n)$ is an $m \times n^2$ array with entries from an n-element set T with the property that the columns of any $2 \times n^2$ subarray consist of all n^2 possible pairs.

The block graph of an orthogonal array $OA(m,n)$, denoted $X_{OA(m,n)}$, is defined to be the graph whose vertices are columns of the orthogonal array, where two columns are adjacent if there exists a row in which they have the same entry.

Let $S_{r,i}$ be the set of columns of $OA(m,n)$ that have the entry i in row r. These sets are cliques, and since each element of the n-element set T occurs exactly n times in each row, the size of $S_{r,i}$ is n for all i and r. These cliques are called the canonical cliques in the block graph $X_{OA(m,n)}$.

A simple combinatorial argument shows that the block graph of an orthogonal array is a strongly regular graph. Moreover, by the Delsarte bound, a clique in $X_{OA(m,n)}$ has size at most n, and the canonical cliques show the tightness of this bound.

Example of an orthogonal array $OA(3, 4)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
1	2	3	4	2	1	4	3	3	4	1	2	4	3	2	1

Canonical cliques:

$$\begin{array}{ccc} \{1,2,3,4\} & \{1,5,9,13\} & \{1,6,11,16\} \\ \{5,6,7,8\} & \{2,6,10,14\} & \{2,5,12,15\} \\ \{9,10,11,12\} & \{3,7,11,15\} & \{3,8,9,14\} \\ \{13,14,15,16\} & \{4,8,12,16\} & \{4,7,10,13\} \end{array}$$

A non-canonical maximum clique: $\{1, 2, 5, 6\}$. Note that these columns form an orthogonal subarray $OA(3, 2)$.

Intersecting columns in an orthogonal array

If we view columns of an orthogonal array that have the same entry in the same row as intersecting columns, then we can view the Delsarte bound as the bound in the EKR theorem for intersecting columns of an orthogonal array. The question is, under what conditions will all cliques of size n in the graph $X_{OA}(m,n)$ be canonical? The following answer can be viewed as the uniqueness part of the EKR theorem.

Theorem 5 ([GM15, Corollary 5.5.3])

Let $X = X_{OA(m,n)}$ be the block graph of an orthogonal array $OA(m,n)$ with $n > (m-1)^2$. Then X has the strict-EKR property: the only maximum cliques in X are the columns that have entry i in row r for some $1 \leq i \leq n$ and $1 \leq r \leq m$.

This is equivalent to saying that the largest set of intersecting columns in an orthogonal array is the set of all columns that have the same entry in the some row, and these sets are the only maximum intersecting sets.

Open problems for the block graphs of orthogonal arrays

Problem 8

Find a characterisation of the orthogonal arrays, based only on the parameters of the array, for which all of the maximum cliques in the orthogonal array graph are canonical cliques.

Problem 9

Assume that $OA(m,(m-1)^2)$ *is an orthogonal array and its orthogonal array graph has non-canonical cliques of size* $(m-1)^2$*. Do these non-canonical cliques form subarrays?*

[GM15, Section 5.5] provides an example of non-canonical cliques in the block graph of an orthogonal array that form subarrays.

Problem 10

Determine all the maximum cliques in the block graph for any orthogonal array.

Identification of the elements of $\mathbb{F}_{q^2}$ and the points of $\mathrm{AG}(2, q)$

A finite field $\mathbb{F}_{q^2}$ can be viewed in a canonical way as a two-dimensional vector space over $\mathbb{F}_q$, or, as the affine plane $\mathrm{AG}(2, q)$.

Each nonzero element uniquely defines a line through 0 and can be viewed as a slope of this line.

Peisert-type graphs

Let q be a prime power. Let $S \subset \mathbb{F}_{q^2}^*$ be a union of $m \leq q$ cosets of $\mathbb{F}_q^*$ in $\mathbb{F}_{q^2}^*$ such that $\mathbb{F}_q^* \subset S$, that is,

$$S = c_1\mathbb{F}_q^* \cup c_2\mathbb{F}_q^* \cup \cdots \cup c_m\mathbb{F}_q^*.$$

Then the Cayley graph $X = \mathrm{Cay}(\mathbb{F}_{q^2}^+, S)$ is said to be a Peisert-type graph of type (m, q). A clique in X is called a canonical clique if it is the image of the subfield $\mathbb{F}_q$ under an affine transformation.

A Peisert-type graph of type (m, q) can be viewed as a graph on the points of the affine geometry $\mathrm{AG}(2, q)$ with two points being adjacent whenever the line through these points belongs to one of m prescribed parallel classes of lines. Peisert-type graphs are equivalent to the block graphs of orthogonal arrays obtained from the parallel classes of the affine plane $\mathrm{AG}(2, q)$.

Further investigation (I)

In [GY23], we proved that the non-canonical cliques (when exist) in the block graphs of orthogonal arrays with parameters $OA(\sqrt{q}+1, q)$ obtained from $\mathrm{AG}(2, q)$ necessarily have the subarray structure.

An interesting project could arise from the investigation of the EKR properties of the block graphs of orthogonal arrays with parameters $OA(\sqrt{q}+1, q)$ obtained from affine planes different from $\mathrm{AG}(2, q)$.

In particular, it is possible to examine the database of small projective planes by Eric Moorhouse (the deletion of a line from projective plane together with its points results in an affine plane that depends on the choice of the deleted line). We have checked many such small orthogonal arrays $OA(\sqrt{n}+1, n)$ and found that all non-canonical cliques in the block graphs have a subarray structure. We thus formulate the following conjecture.

[GY23] S. Goryainov, C. H. Yip, *Extremal Peisert-type graphs without the strict-EKR property*, June 2023, `https://arxiv.org/abs/2306.00391`

Further investigation (II)

Conjecture 1

Let X be the block graph of an orthogonal array $OA(\sqrt{n}+1, n)$, having a non-canonical maximum clique. Then all its non-canonical maximum cliques have a subarray structure.

Concluding remarks

In this lecture we have discussed some details of the extension of the EKR theorem to two important class of strongly regular graphs: the block graphs of 2-$(n, m, 1)$ designs and the block graphs of orthogonal arrays.

Thank you for your attention!