On finite generalized quadrangles with $\mathrm{PSL}(2, q)$ as an automorphism group

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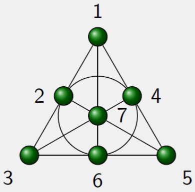
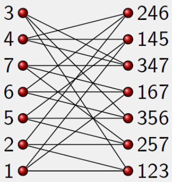
Generalized polygons

- In 1959 Tits introduced the concept of generalized polygons in order to study the simple groups of Lie type systematically, and his work builds a bridge between geometry and group theory.
- A finite generalized n-gon is a finite point-line incidence geometry whose bipartite incidence graph has diameter n and girth $2n$. It is *thick* if each line contains at least three points and each point is on at least three lines.
- The Feit-Higman theorem shows that finite thick generalized n-gons exist only for $n = 2, 3, 4, 6$ or 8. A finite generalized 3-gon is a projective plane, and a finite generalized 4-gon is also called a generalized quadrangle.

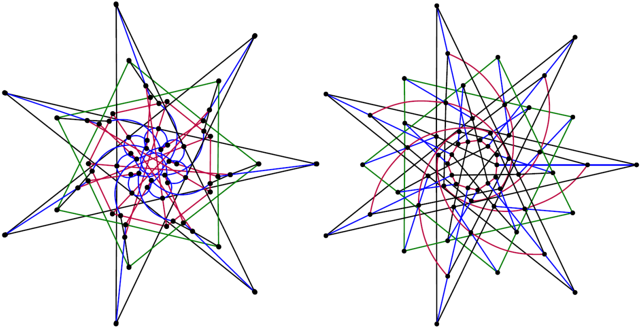
Generalized polygons





- The Fano plane is a projective plane of order 2.

Generalized polygons



- The two generalized hexagons of order 2. Each is the point – line dual of the other.

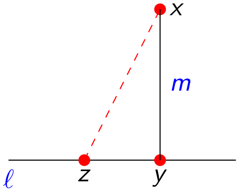
Generalized quadrangles

Definition

A **finite generalized quadrangle** (GQ) $\mathcal{S}$ of order (s, t) is a point-line incidence geometry $(\mathcal{P}, \mathcal{L}, I)$, where

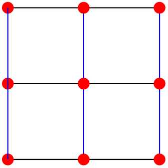
- each point is incident with $t + 1$ lines, every two points are incident with at most one line;
- each line is incident with $s + 1$ points;
- GQ Axiom: for each point-line pair (x, ℓ) that is not incident there is exactly one point y on ℓ that is collinear with x.

- GQ Axiom $\Rightarrow$ no triangles

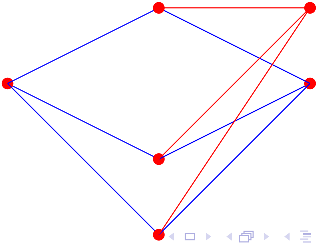


- $t = 1$, a grid;
- $s = 1$, a dual grid;

grid:



dual grid:



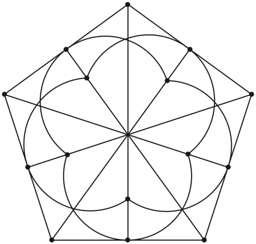
Classical examples

TABLE 1. The classical generalized quadrangles given by certain rank 3 classical groups.

$\mathcal{Q}$	Order	soc(G)	Point stabilizer in soc(G)
$\mathrm{W}(3,q)$, q odd	(q,q)	$\mathrm{PSp}_4(q)$	$E_q^{1+2}:(\mathrm{GL}_1(q)\circ\mathrm{Sp}_2(q))$
$\mathrm{W}(3,q)$, q even	(q,q)	$\mathrm{Sp}_4(q)$	$E_q^3:\mathrm{GL}_2(q)$
$\mathrm{Q}(4,q)$, q odd	(q,q)	$\mathrm{P\Omega}_5(q)$	$E_q^3:((\frac{(q-1)}{2}\times\Omega_3(q)).2)$
$\mathrm{Q}^-(5,q)$	(q,q^2)	$\mathrm{P\Omega}_6^-(q)$	$E_q^4:(\frac{q-1}{\lvert Z(\Omega_6^-(q))\rvert}\times\Omega_4^-(q))$
$\mathrm{H}(3,q^2)$	(q^2,q)	$\mathrm{PSU}_4(q)$	$E_q^{1+4}:\left(\mathrm{SU}_2(q):\frac{q^2-1}{\gcd(q+1,4)}\right)$
$\mathrm{H}(4,q^2)$	(q^2,q^3)	$\mathrm{PSU}_5(q)$	$E_q^{1+6}:\left(\mathrm{SU}_3(q):\frac{q^2-1}{\gcd(q+1,5)}\right)$
$\mathrm{H}(4,q^2)^D$	(q^3,q^2)	$\mathrm{PSU}_5(q)$	$E_q^{4+4}:\mathrm{GL}_2(q^2)$

- The classical generalized quadrangles come in dual pairs: $W(3,q)$ is isomorphic to the dual of $Q(4,q)$, $Q^-(5,q)$ is isomorphic to the dual of $H(3,q^2)$, and $H(4,q^2)^D$ denotes the dual of $H(4,q^2)$.

Unique GQ of order 2: $W(2)$



- The points of $\mathrm{PG}(3,2)$, together with the totally isotropic lines with respect to the alternating form $b(x,y) = x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3$ of $\mathrm{PG}(3,2)$, form the $\mathrm{GQ}(2,2)$.

Lemma

Let $\mathcal{S}$ be a finite thick generalized quadrangle of order (s,t) with point set $\mathcal{P}$ and line set $\mathcal{L}$. Then $|\mathcal{P}| = (s+1)(st+1)$, $|\mathcal{L}| = (t+1)(st+1)$, and the following properties hold:

(i) *(Higman's inequality) $s \leq t^2$ and $t \leq s^2$;*

(ii) *(Divisibility condition) $s+t$ divides $st(s+1)(t+1)$;*

The automorphism groups of GQ

An **automorphism** of a GQ is a permutation of the points and lines, which preserves the incidence. A **flag** is an incident point-line pair. A GQ which admits an automorphism group acting transitively on the set of flags is called flag-transitive.

Conjecture (Kantor, 1991)

A finite flag-transitive GQ *is classical, the unique* $\mathrm{GQ}(3,5)$ *or the generalized quadrangle of order (15,17) arising from the Lunelli-Sce hyperoval up to duality.*

Antiflag-transitive and locally 2-transitive generalized quadrangles

Theorem (Bamberg, Li, and Swartz, 2017)

Let $\mathcal{S}$ be a finite thick generalized quadrangle and suppose G is a subgroup of automorphisms of $\mathcal{S}$ acting transitively on the antiflags (nonincident point-line pairs) of $\mathcal{S}$. Then $\mathcal{S}$ is isomorphic to a classical generalized quadrangle or to the unique $\mathrm{GQ}(3,5)$ or its dual.

Theorem (Bamberg, Li, and Swartz, 2020)

If $\mathcal{S}$ is a finite thick locally 2-transitive (transitive on ordered pairs of collinear points and ordered pairs of concurrent lines) generalized quadrangle, then $\mathcal{S}$ is isomorphic to a classical generalized quadrangle or to the unique $\mathrm{GQ}(3,5)$ or its dual.

Point-primitive and line-primitive generalized quadrangles

G is primitive on Ω if it is transitive and there is no non-trivial equivalence relation on Ω which is G-invariant: equivalently, if the stabilizer G_α of a point $\alpha \in \Omega$ is a maximal subgroup of G.

Theorem (Bamberg et al., 2012)

A group of automorphisms acting primitively on the points and lines of a finite thick generalized quadrangle is almost simple. Let G be an almost simple group of automorphisms of a finite thick generalized quadrangle $\mathcal{S}$.

- *If G acts primitively on the points and lines of $\mathcal{S}$, then the socle of G is not a sporadic simple group.*
- *If G acts flag-transitively and point-primitively on $\mathcal{S}$ and the socle of G is an alternating group A_n with $n \geqslant 5$, then $G \leqslant \mathrm{S}_6$ and $\mathcal{S}$ is the unique generalized quadrangle of order $(2, 2)$.*

Maximal subgroups of the almost simple groups with socle $\mathrm{PSL}(2, q)$

Lemma (Giudici, 2007)

Let G be an almost simple group with socle $X = \mathrm{PSL}(2, q)$, where $q = p^f \geqslant 4$ for a prime p. Let M be a maximal subgroup of G not containing X, and set $M_0 := M \cap X$. Then M_0 is a maximal subgroup of X as listed in Table 1 with some exceptions.

Maximal subgroups of $\mathrm{PSL}(2, q)$

Table: Maximal subgroups of $X = \mathrm{PSL}(2, q)$ and their indices in X

Case	M_0	$[X : M_0]$
1	$\mathrm{C}_p^f \rtimes \mathrm{C}_{\frac{q-1}{\gcd(2,q-1)}}$	$q+1$
2	$\mathrm{PGL}(2, q_0)$	$\frac{q_0(q_0^2+1)}{2}$
3	A_5	$\frac{q(q^2-1)}{120}$
4	A_4	$\frac{p(p^2-1)}{24}$
5	S_4	$\frac{p(p^2-1)}{48}$
6	$\mathrm{PSL}(2, q_0)$	$\frac{q_0^{r-1}(q_0^{2r}-1)}{q_0^2-1}$
7	$\mathrm{PGL}(2, q_0)$	$\frac{q_0^{r-1}(q_0^{2r}-1)}{q_0^2-1}$
8	$\mathrm{D}_{2(q-1)/\gcd(2,q-1)}$	$\frac{q(q+1)}{2}$
9	$\mathrm{D}_{2(q+1)/\gcd(2,q-1)}$	$\frac{q(q-1)}{2}$

M_0, M_1 have distinct case numberings in Table 1

Let $\mathcal{S}$ be a finite think generalized quadrangle of order (s, t) with point set $\mathcal{P}$ and line set $\mathcal{L}$, and suppose that G is an automorphism group of $\mathcal{S}$ that acts primitively on both $\mathcal{P}$ and $\mathcal{L}$. Fix a point α and a line ℓ of $\mathcal{S}$, and set

$$M_0 := G_\alpha \cap X, \quad M_1 := G_\ell \cap X.$$

Since X is normal in G, it is transitive on both $\mathcal{P}$ and $\mathcal{L}$ by the primitivity assumption. We thus have

$$|\mathcal{P}| =(s+1)(st+1) = \frac{|X|}{|M_0|}, \tag{1}$$

$$|\mathcal{L}| =(t+1)(st+1) = \frac{|X|}{|M_1|}. \tag{2}$$

A coset geometry model

We define a set D as follows:

$$D = \{g \in G : \alpha^g \text{ is incident with } \ell\}.$$

The points on the line ℓ are α^g for $g \in D$, and the lines through the point α are $\ell^{g^{-1}}$ for $g \in D$. We thus have $|D| = (s+1)|G_\alpha| = (t+1)|G_\ell|$. The set D is a union of (G_α, G_l)-double cosets in G, so we have a decomposition

$D = \bigcup_{i=1}^{d} G_\alpha h_i G_l$, where the double cosets $G_\alpha h_i G_l$, $1 \le i \le d$, are pairwise distinct.

It follows that

$$s+1 = \frac{|D|}{|G_\alpha|} = \sum_{i=1}^{d} \frac{|G_l|}{|G_l \cap h_i^{-1} G_\alpha h_i|},$$

$$t+1 = \frac{|D|}{|G_\ell|} = \sum_{i=1}^{d} \frac{|G_\alpha|}{|G_\alpha \cap h_i G_l h_i^{-1}|}.$$

A coset geometry model

Theorem

Suppose that G is a finite group, and H, K are its subgroups such that $[G:H]=(1+s)(1+st)$, $[G:K]=(1+t)(1+st)$ for some integers $s,t\geq 2$. Let D be a union of (H,K)-double cosets of G with $|D|=(s+1)|H|$. We define an incidence relation $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ as follows: $\mathcal{P}$ is the set of right cosets of H in G, $\mathcal{L}$ is the set of right cosets of K, and a point Hg_1 is incident with a line Kg_2 if and only if $g_1g_2^{-1}\in D$. Then $\mathcal{S}$ is a generalized quadrangle if and only if the following conditions hold:

- (i) *For each $g\in G\setminus D$, there exist three elements $g_1,g_2,g_3\in D$ such that $g=g_1g_2^{-1}g_3$, and if $g=g_1'g_2'^{-1}g_3'$ is another such expression, then $g_1'=g_1u$, $g_2'=vg_2u$ and $g_3'=vg_3$ for some $u\in K$ and $v\in H$.*
- (ii) *If $g\in D$ and $g=g_1g_2^{-1}g_3$ for some $g_1,g_2,g_3\in D$, then either $g_1g_2^{-1}\in H$ or $g_2^{-1}g_3\in K$.*

M_0, M_1 have the same case numberings in Table 1

Lemma

If $M_0 \cong \mathrm{PGL}(2, q_0)$ with $q = q_0^2$ odd, then $q = 9$ and $\mathcal{S}$ is $W(2)$.

Lemma

Let $\mathcal{P}_g$ be the set of fixed points of g, and suppose that α is in $\mathcal{P}_g$. If $[C_G(g) : C_G(g) \cap G_\alpha] = |\mathcal{P}_g|$, then $C_G(g)$ acts transitively on $\mathcal{P}_g$.

Lemma

Let G be a finite transitive permutation group on a set Ω, and choose $\alpha \in \Omega$. For $g \in G$, let g^G be its conjugacy class in G and let $Fix(g)$ be its number of fixed points on Ω. We have

$$Fix(g) = \frac{|\Omega| \cdot |g^G \cap G_\alpha|}{|g^G|}.$$

Conjugacy classes of involutions in $\mathrm{PSL}(2, q)$

Lemma

Suppose that $X = \mathrm{PSL}(2, q)$, $q = p^f \geq 4$ with p prime. Then X has a single conjugacy class C of involutions and

$$|C| = \begin{cases} q^2 - 1, & \text{if } q \text{ is even,} \\ \frac{1}{2}q(q+\epsilon), & \text{if } q \equiv \epsilon \pmod 4 \text{ with } \epsilon \in \{\pm 1\}. \end{cases}$$

Moreover, if g is an involution in X, then

$$C_X(g) = \begin{cases} \mathrm{D}_{q-1}, & \text{if } q \equiv 1 \pmod 4, \\ \mathrm{D}_{q+1}, & \text{if } q \equiv 3 \pmod 4, \\ \mathrm{C}_2^f, & \text{if } q = 2^f,\ f \geq 2. \end{cases}$$

Conjugacy classes of elements of order 3 in $\mathrm{PSL}(2, q)$

Lemma

Suppose that $X = \mathrm{PSL}(2, q)$, $q = p^f$ with $p > 5$ prime. Then X has a single conjugacy class C of elements of order 3 in X and

$$|C| = \begin{cases} q(q-1), & \textit{if} \;\; q \equiv -1 \pmod 3; \\ q(q+1), & \textit{if} \;\; q \equiv 1 \pmod 3. \end{cases}$$

Moreover, if g is an element of order 3 in X, then

$$C_X(g) = \begin{cases} \mathrm{C}_{(q-1)/2}, & \text{if } q \equiv 1 \pmod 3, \\ \mathrm{C}_{(q+1)/2}, & \text{if } q \equiv 2 \pmod 3. \end{cases}$$

Fixed substructure of generalized quadrangles

Theorem

Let g be an automorphism of a generalized quadrangle $\mathcal{S} = (\mathcal{P}, \mathcal{L})$. Let $\mathcal{P}_g$ and $\mathcal{L}_g$ be the set of fixed points and fixed lines of g respectively, and let $\mathcal{S}_g = (\mathcal{P}_g, \mathcal{L}_g)$ be the fixed substructure. Then one of the following holds:

- $\mathcal{P}_g = \mathcal{L}_g = \varnothing$,
- $\mathcal{L}_g = \varnothing$, $\mathcal{P}_g$ *is a nonempty set of pairwise noncollinear points,*
- $\mathcal{P}_g = \varnothing$, $\mathcal{L}_g$ *is a nonempty set of pairwise nonconcurrent lines,*
- $\mathcal{L}_g$ *is nonempty, and* $\mathcal{P}_g$ *contains a point* P *that is collinear with each point of* $\mathcal{P}_g$ *and is on each line of* $\mathcal{L}_g$,
- $\mathcal{P}_g$ *is nonempty, and* $\mathcal{L}_g$ *contains a line* ℓ *that is concurrent with each line of* $\mathcal{L}_g$ *and contains each point of* $\mathcal{P}_g$,
- $\mathcal{S}_g$ *is a grid with parameters* (s_1, s_2), $s_1 < s_2$,
- $\mathcal{S}_g$ *is a dual grid with parameters* (s_1, s_2), $s_1 < s_2$,
- $\mathcal{S}_g$ *is a generalized quadrangle of order* (s', t').

Fixed substructure of generalized quadrangles

Corollary

With the same notation, we have the following properties:

(i) *If $|\mathcal{P}_g| \geq 2$, $|\mathcal{L}_g| \geq 2$ and $\mathcal{S}_g$ admits an automorphism group H that is transitive on both points and lines, then $\mathcal{S}_g$ is a subquadrangle.*

(ii) *If $|\mathcal{P}_g| = |\mathcal{L}_g| \geq 2$ and $\mathcal{S}_g$ admits an automorphism group H that is transitive on its points, then $\mathcal{S}_g$ is a subquadrangle.*

Lemma

If G is a finite group acting regularly on the points of a finite thick generalized quadrangle of order s, then it is nonabelian.

Lemma

Let $\mathcal{S}$ be a finite thick generalized quadrangle of order (s, t). Then there is no abelian group that acts transitively on both the points and the lines of $\mathcal{S}$.

On finite generalized quadrangles with $\mathrm{PSL}(2, q)$ as an automorphism group

Theorem (Feng, Lu, 2023)

Suppose that G is an automorphism group of a finite thick generalized quadrangle $\mathcal{S}$ that is primitive on both points and lines. If G is an almost simple group with socle $\mathrm{PSL}(2, q)$, $q \geq 4$, then $q = 9$ and $\mathcal{S}$ is the symplectic quadrangle $W(2)$.

On finite generalized quadrangles with $\mathrm{PSU}(3, q)$ as an automorphism group

Theorem (Lu, Zhang, Zou, 2023+)

Let G be an automorphism group of a finite thick generalized quadrangle $\mathcal{S}$. If G acts primitively on both points and lines of $\mathcal{S}$, then the socle of G cannot be $\mathrm{PSU}(3, q)$ with $q \geq 3$.

Current progress

- point-primitive+line-primitive $\Rightarrow$ G must be AS type; (Bamberg et al., 2012)
- point-primitive+line-transitive+HA type$\Rightarrow$ GQ$(3,5)$, LSce$(15,17)$ (Bamberg et al., 2016)
- point-primitive $\Rightarrow$ G cannot be HS, HC type; (Bamberg et al., 2017) (Di, Feng, 2023+)

The classification of flag-transitive generalized quadrangles

Questions:

- Is it possible to classify all point-primitive and line-primitive generalized quadrangles? If add the condition of flag-transitive?
- G is a point-primitive automorphism group of a GQ $\Rightarrow$ line-transitive or has a hemisystem? (Bamberg, Evans, 2021)
- G is a point-primitive automorphism group of a GQ $\Rightarrow$ HA or AS type?

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Thanks for your attention!