Card Shuffle Groups

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This is joint work with Binzhou Xia and Zhishuo Zhang.

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Perfectly shuffle $2n$ cards

- Cut the deck in half:

$$\begin{array}{r} 0\,\text{—} \\ 1\,\text{—} \\ \vdots \\ 2n-1\,\text{—} \end{array} \quad \longrightarrow \quad \begin{array}{rl} 0\,\text{—} & \text{—}\,n \\ 1\,\text{—} & \text{—}\,n+1 \\ \vdots & \vdots \\ n-1\,\text{—} & \text{—}\,2n-1 \end{array}$$

- Perfectly interleave them:

$$\begin{array}{rl} 0\,\text{—} & \\ & \text{—}\,n \\ 1\,\text{—} & \\ & \text{—}\,n+1 \\ \vdots & \\ n-1\,\text{—} & \\ & \text{—}\,2n-1 \end{array} \qquad\qquad \begin{array}{rl} & \text{—}\,n \\ 0\,\text{—} & \\ & \text{—}\,n+1 \\ 1\,\text{—} & \\ \vdots & \\ & \text{—}\,2n-1 \\ n-1\,\text{—} & \end{array}$$

Out-shuffle O In-shuffle I

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Question: Can it return to the original order?

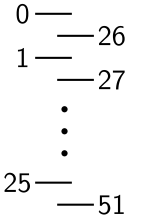
Answer: Yes. For example, after 52! times.

Question: What is the minimum number of times needed to return to the original order?

Answer: 8 times.

Why 8 times

- Position x: 0 1 2 3 4 $\cdots$ 25 26 $\cdots$ 50 51



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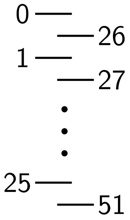

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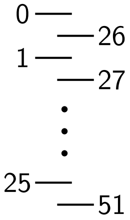

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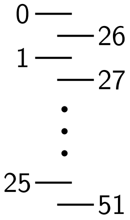

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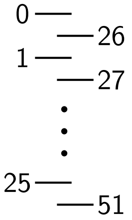

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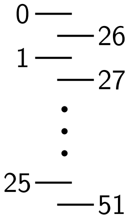

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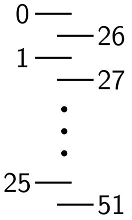

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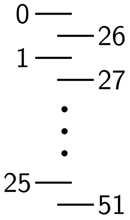

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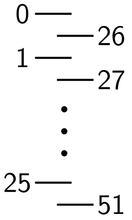

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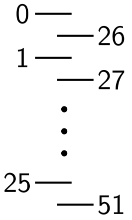

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To answer these questions, we first determine the parity of O and I.

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and thus its inversion number is $1 + \cdots + n - 1 = n(n - 1)/2$.

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- Thus $\langle O, I \rangle \leq \mathrm{Alt}(2n) \Longleftrightarrow n \equiv 0 \pmod 4$.

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For a general n, out-shuffle and in-shuffle preserve the partition $\{0, 2n-1\}, \{1, 2n-2\}, \ldots, \{n-1, n\}$.

$\langle O, I \rangle$ is neither $\mathrm{Alt}(2n)$ nor $\mathrm{Sym}(2n)$

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Diaconis-Graham-Kantor

Question: How many/Which orderings can be obtained by performing a sequence of the two shuffles?

[1] P. Diaconis, R. L. Graham and W. M. Kantor, The mathematics of perfect shuffles., *Adv. Appl. Math.*, 4 (1983), 175–196.

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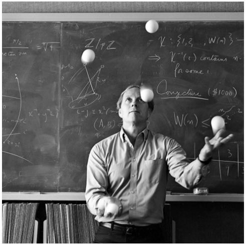
Answered by Diaconis, Graham and Kantor in 1983[1].

Persi Diaconis

ICM talk in 1990



Ron Graham

ICM talk in 1983

William M. Kantor

ICM talk in 1998

[1] P. Diaconis, R. L. Graham and W. M. Kantor, The mathematics of perfect shuffles., *Adv. Appl. Math.*, 4 (1983), 175–196.

The classification of $\langle O, I\rangle$ in [1]

Size of each pile n	$\langle O, I\rangle$
$n = 2^f$ for some positive integer f	$C_2 \wr C_{f+1}$
$n \equiv 0 \pmod 4$, $n > 12$ and n is not a power of 2	$C_2^{n-1} \rtimes A_n$
$n \equiv 1 \pmod 4$	$C_2^n \rtimes A_n$
$n \equiv 2 \pmod 4$ and $n > 6$	$C_2 \wr \mathrm{Sym}(n)$
$n \equiv 3 \pmod 4$	$C_2^{n-1} \rtimes S_n$
$n = 6$	$C_2^6 \rtimes \mathrm{PGL}(2,5)$
$n = 12$	$C_2^{11} \rtimes M_{12}$

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A deck of kn cards with $k \geq 2$

- cut into k piles and then perfectly interleave them ($k!$ ways).

$$\begin{matrix} 0 \\ 1 \\ \vdots \\ kn-1 \end{matrix} \quad \longrightarrow \quad \begin{matrix} 0 & n & \cdots & (k-1)n \\ 1 & 1+n & \cdots & 1+(k-1)n \\ \vdots & \vdots & & \vdots \\ n-1 & 2n-1 & \cdots & kn-1 \end{matrix}$$

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- ρ_τ: the permutation of the kn cards induced by the permutation τ of the k piles.

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- In [2] they also proved:
  - $G_{k,kn} \leq \mathrm{Alt}(kn)$ if and only if either $n \equiv 0 \pmod 4$, or $n \equiv 2 \pmod 4$ and $k \equiv 0$ or $1 \pmod 4$.
  - $G_{k,k^m} = \mathrm{Sym}(k) \wr C_m$.

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Shuffle Group Conjecture (2005)

For $k \geq 3$, if n is not a power of k and $(k, n) \neq (4, 2^f)$ for any positive integer f, then $G_{k,kn} \geq A_{kn}$.

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Our contribution

Theorem (Xia-Zhang-Z. 2023^{+})

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- Reduction Lemma: If $G_{k,kn}$ is 2-transitive, then either $k = 4$ and n is a power of 2, or $G_{k,kn}$ contains A_{kn}.
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- A permutation group G on a set Ω is said to be 2-transitive if the induced action of G on $\Omega \times \Omega \setminus \{(\alpha, \alpha) \mid \alpha \in \Omega\}$ is transitive.

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Write $G = G_{k,kn}$. Suppose G is an affine 2-transitive permutation group, i.e., $G \leq \mathrm{AGL}(V) = \mathrm{AGL}(d, p)$ for some d-dimension vector space V over a prime field $\mathbb{F}_p$.

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- Note that $G_{3,3^d}$ and $G_{4,2^d}$ have been determined.

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2-transitivity Lemma (Xia-Zhang-Z. 2023^{+})

If either $k \geq 4$ and n is not a power of k, or $k = 3$ and n is not divisible by 3, then $G_{k,kn}$ is 2-transitive.

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- Shuffle Group Conjecture $\xrightarrow{\textit{Reduction Lemma}}$ 2-transitivity of $G_{k,kn}$.
- Thus the remaining unresolved case of the conjecture is that $k = 3$ divides n.

Sketch of Proof to 2-transitivity Lemma

- Let $G = G_{k,kn}$ and $[m] = \{0, 1, \ldots, m-1\}$.

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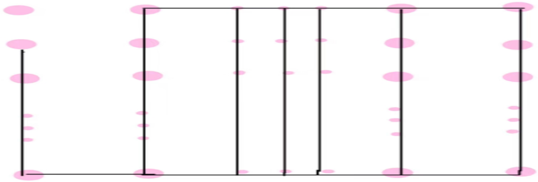

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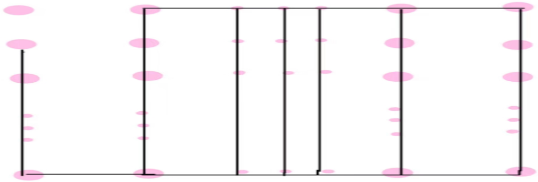

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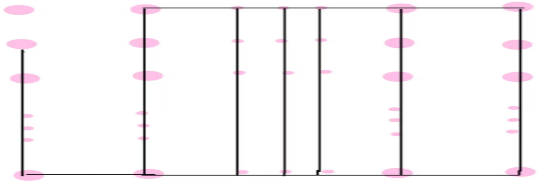

- we complete this proof through the following cases:
  - $k \nmid n$ with $k \geq 4$; (only elaborate on this case here)
  - $k \nmid n$ with $k = 3$;
  - $k \mid n$ with $k \geq 4$.

Case for $k \nmid n$ with $k \geq 4$

$$\begin{array}{cccccc}
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\cdot & \cdot & & & & \cdot \\
\cdot & \cdot & & & & \cdot \\
\cdot & \cdot & & & & \cdot \\
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- $(i+jn)^{\rho_\tau} = i + j^\tau n$;

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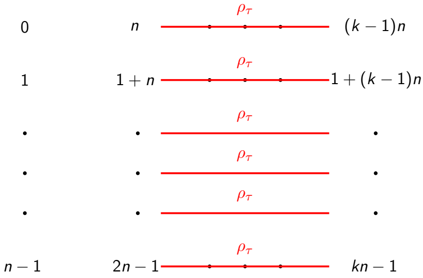

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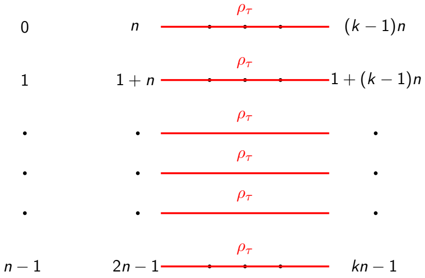

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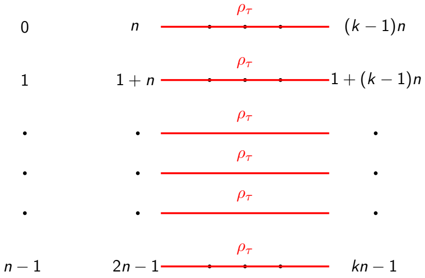

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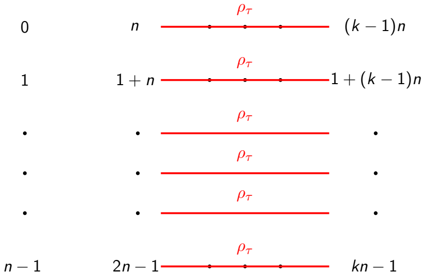

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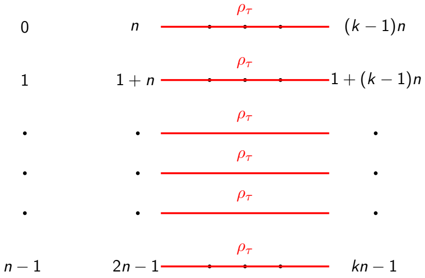

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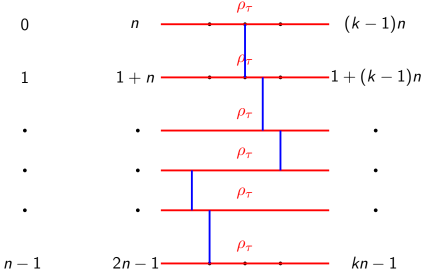

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  $\Longrightarrow \{ik+1, ik+2\ldots, ik+k-1\}$ in the same orbit of G_0.
- $k \geq 4$ and $k \nmid n \implies$ for $\forall\ x \in [n]$, $\exists\ y \in \{x+n, x+2n, x+3n\}$ s.t. $y \sim y+1$.

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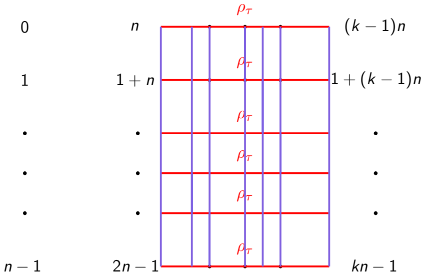
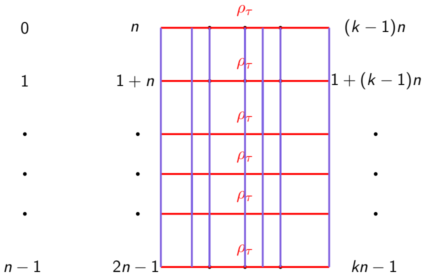


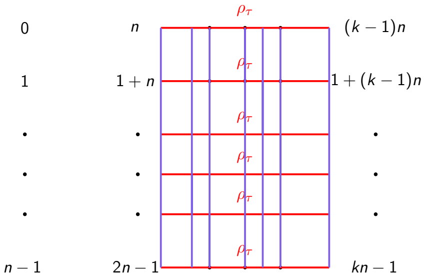
- $(i + jn)^{\rho_\tau} = i + j^\tau n$;
- recall $(i + jn)^\sigma = ik + j$;
- $(ik + 1)^{\sigma^{-1}\rho_\tau\sigma} = (i + 1 \cdot n)^{\rho_\tau\sigma} = (i + 1^\tau \cdot n)^\sigma = ik + 1^\tau$
  $\Longrightarrow \{ik + 1, ik + 2 \ldots, ik + k - 1\}$ in the same orbit of G_0.
- $k \geq 4$ and $k \nmid n \Longrightarrow$ for $\forall\ x \in [n]$, $\exists\ y \in \{x + n, x + 2n, x + 3n\}$ s.t. $y \sim y + 1$.

Case for $k \nmid n$ with $k \geq 4$



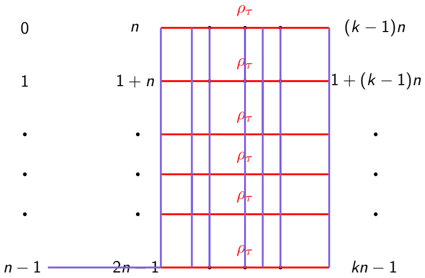
- $x^\sigma \equiv kx \pmod{kn-1}$ for all $x \in [kn]$;

Case for $k \nmid n$ with $k \geq 4$



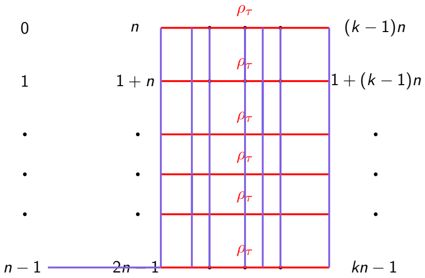
- $x^\sigma \equiv kx \pmod{kn-1}$ for all $x \in [kn]$;
- $(n-1)^\sigma = k(n-1)$

Case for $k \nmid n$ with $k \geq 4$



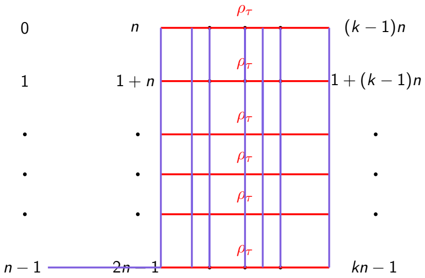
- $x^\sigma \equiv kx \pmod{kn-1}$ for all $x \in [kn]$;
- $(n-1)^\sigma = k(n-1) \xrightarrow{n-1<(n-1)k<kn-1} n-1 \in n^{G_0}$;

Case for $k \nmid n$ with $k \geq 4$



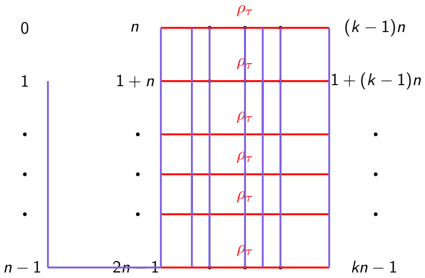
- $x^\sigma \equiv kx \pmod{kn-1}$ for all $x \in [kn]$;
- $(n-1)^\sigma = k(n-1) \xrightarrow{n-1<(n-1)k<kn-1} n-1 \in n^{G_0}$;
- suppose $\exists\ x \in [n-1] \backslash \{0\}$ s.t $y \in n^{G_0}$ for all $y > x$.

Case for $k \nmid n$ with $k \geq 4$



- $x^\sigma \equiv kx \pmod{kn-1}$ for all $x \in [kn]$;
- $(n-1)^\sigma = k(n-1) \xrightarrow{n-1<(n-1)k<kn-1} n-1 \in n^{G_0}$;
- suppose $\exists\ x \in [n-1] \backslash \{0\}$ s.t $y \in n^{G_0}$ for all $y > x$.
- $x^\sigma = kx > x$

Case for $k \nmid n$ with $k \geq 4$



- $x^\sigma \equiv kx \pmod{kn-1}$ for all $x \in [kn]$;
- $(n-1)^\sigma = k(n-1) \xrightarrow{n-1<(n-1)k<kn-1} n-1 \in n^{G_0}$;
- suppose $\exists\ x \in [n-1]\backslash\{0\}$ s.t $y \in n^{G_0}$ for all $y > x$.
- $x^\sigma = kx > x \implies x \in n^{G_0}$.

Thank you for listening!