Orientably-Regular p-Maps and Regular p-Maps

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2023-10-17

Algebraic Maps

Definition

For a given finite set F and three fixed-point-free involutory permutations t, r, ℓ on F, a quadruple $\mathcal{M} = \mathcal{M}(F; t, r, \ell)$ is called a *combinatorial map* if they satisfy two conditions: (1) $t\ell = \ell t$; (2) the group $\langle t, r, \ell \rangle$ acts transitively on F.

Definition

For a given finite set D and two fixed-point-free permutations r, ℓ on D where ℓ is an involutory, a triple $\mathcal{M} = \mathcal{M}(D; r, \ell)$ is called a *combinatorial orientable map* if $\langle r, \ell \rangle$ acts transitively on D.

(1) **Regular Maps.**

A regular map can be constructed in the following way.
Let G be any finite group that is generated by three involutions t, r and ℓ such that $t\ell$ has order 2, and tr and $r\ell$ have order at least 2. The elements of G are taken as the set of flags of $\mathcal{M}$. The left multiplication of t, r and ℓ on the set of flags are called *transversal*, *rotary* and *longitudinal* involution respectively. We take the vertices, edges and face-boundaries by the right cosets of $\langle r, t\rangle$, $\langle t, \ell\rangle$ and $\langle r, \ell\rangle$ in G, respectively, with incidence given by non-empty intersection. We denote such a map by $\mathcal{M}(G; r, t, \ell)$.

Proposition

Given an abstract group G and two triples of generators (r_1, t_1, ℓ_1) and (r_2, t_2, ℓ_2),

$$\mathcal{M}_1 = \mathcal{M}(G; r_1, t_1, \ell_1) \cong \mathcal{M}_2 = \mathcal{M}_2(G; r_2, t_2, \ell_2)$$

if and only if

$$r_1^\sigma = r_2, t_1^\sigma = t_2, l_1^\sigma = l_2$$

for some σ in $Aut(G)$.

(2) **Orientably-regular maps.**

An orientably-regular map can be constructed in the following direct way.
Let G be any finite group that is generated by an element r of order at least 2 and an involution ℓ. The elements of G are taken as the set of arcs of $\mathcal{M}$. We take vertices, edges, face-boundaries by right cosets of $\langle r \rangle$, $\langle \ell \rangle$ and $\langle r\ell \rangle$ in G, respectively, with incidence given by non-empty intersection of cosets. The left multiplication of r and ℓ on the arcs are called *local rotation* and *arc-revision involution* respectively. We denote such a map by $\mathcal{M}(G; r, \ell)$.

Moreover, two maps $\mathcal{M}(G; r_1, \ell_1) \cong \mathcal{M}(G; r_2, \ell_2)$ if and only if there exists an automorphism σ of G such that $r_1^\sigma = r_2$ and $\ell_1^\sigma = \ell_2$.

Definition (p-map)

A map is called a *p-map* if the number of vertices is p^k, where p is prime and $k \geq 1$.

Definition (π-map)

We call the map $\mathcal{M}$ of n vertices a π*-map* if all prime divisors of n lie on π.

Recall that for a given set π of primes, π' means the set of all primes not containing in π and an integer is called a π-number if all of its prime divisors lie in π. Let G be a finite group, a π-subgroup refers to a subgroup whose order is a π-number and a Hall π-subgroup is meant a π-subgroup whose index in G is a π'-number. The maximal normal subgroup of odd order is denoted by $O_{2'}(G)$.

Definition

- An orientably-regular (resp. A regular) *p*-map $\mathcal{M}$ is called *solvable* if $\mathrm{Aut}^+(\mathcal{M})$ (resp. $\mathrm{Aut}(\mathcal{M})$) is solvable; and called *normal* if $\mathrm{Aut}^+(\mathcal{M})$ (resp. $\mathrm{Aut}(\mathcal{M})$) contains the normal Sylow *p*-subgroup.

Definition

- An orientably-regular $($resp. A regular $)$ π-map $\mathcal{M}$ is called *solvable* if $\mathrm{Aut}^+(\mathcal{M})$ $($resp. $\mathrm{Aut}(\mathcal{M}))$ is solvable; and $\mathcal{M}$ is called *normal* if $\mathrm{Aut}^+(\mathcal{M})$ $($resp. $\mathrm{Aut}(\mathcal{M}))$ contains a normal Hall π-subgroup, otherwise, $\mathcal{M}$ is called abnormal. Moreover, if $\mathrm{Aut}^+(\mathcal{M})$ $($resp. $\mathrm{Aut}(\mathcal{M}))$ acts primitively on vertices, then we call $\mathcal{M}$ a primitive orientably-regular π-map $($resp. a primitive regular π-map$)$.

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Proposition

Let Γ be a connected simple graph of order p^3 where p is prime and let $\mathcal{M}$ be an orientably-regular embedding of Γ with the group $G = \langle r, \ell \rangle$ of all orientation-preserving automorphisms, where ℓ is an involution and $\langle r \rangle = G_v$ for a vertex v in $V(G)$. Take a Sylow p-subgroup P of G. Then we have the Sylow p-subgroup P is normal in G.

Zhu, Y.H., Du, S.F.: Nonorientable regular embeddings of graphs of order p^3. Journal of Algebraic Combinatorics. 55, 1251–1264 (2022).

Proposition

Let Γ be a connected simple graph of order p^3 and valency n, where p is a prime, and let $\mathcal{M}$ be a nonorientable regular embedding of Γ with automorphism group G. Let N be a minimal normal subgroup of G which induces blocks of minimal size and let K be the kernel of G on the corresponding complete block system. Take a Sylow p-subgroup P of G. Then, we have tne Sylow p-subgroup P is normal in G.

Theorem

Let $\mathcal{M}$ be an orientably-regular p-map or a regular p-map. Then

(1) *$\mathcal{M}$ is solvable;*

(2) *$\mathcal{M}$ is normal, except for the following two cases:*

  (2.1) *$p = 2$, $G/O_2(G) \cong \mathbb{Z}_m \rtimes \mathbb{Z}_2$ or $\mathbb{Z}_m \rtimes \mathbb{D}_4$, where $m \geq 3$ is odd.*

  (2.2) *$p = 3$, $G/O_3(G) \cong S_4$.*

Theorem

Suppose that $\mathcal{M}$ is a nonnormal orientably-regular p-map or a nonnormal regular p-map. Let $G = \mathrm{Aut}^+(\mathcal{M})$ or $\mathrm{Aut}(\mathcal{M})$. Then the quotient map $\overline{\mathcal{M}}$ induced by $O_p(G)$ is one of the following maps:

(1) $p = 2$, $\overline{\mathcal{M}} = \mathcal{D}(m, e)$, *where* $m \geq 3$ *is odd and* $e^2 \equiv 1 (\mod m)$ *but* $e \not\equiv 1 (\mod m)$. $\mathcal{M}$ *is a nonnormal orientably-regular 2-map, more precisely, it is either chiral or reflexible and nonnormal regular;*

(2) $p = 2$, $\overline{\mathcal{M}} = DM(m)$ *and* $\mathcal{M}$ *is nonorientable and nonnormal regular;*

(3) $p = 2$, $\overline{\mathcal{M}} = EM(m)$ *and* $\mathcal{M}$ *is normal orientably-regular but nonnormal regular;*

(4) $p = 3$, $\overline{\mathcal{M}} = \mathcal{C}(3, 2)$ *and* $\mathcal{M}$ *is nonorientable and nonnormal regular.*

solvability of p-map

Lemma

P acts transitively on V so that $G = PG_v = PH$.

Lemma

Let G be a group having a cyclic subgroup H of index a p-power. Then G is solvable.

Proposition

[VI. Hauptsatz 4.3] Let $G = N_1 N_2 \cdots N_k$, where N_i is a nilpotent subgroup of G for all $i \in \{1, 2, \ldots, k\}$, and $N_i N_j = N_j N_i$ for any i, j. Then G is solvable.

B. Huppert, *Endliche Gruppen I*, Springer, Berlin, 1979.

Lemma

Let G be a group having a subgroup $H \cong \mathbb{D}_{2n}$ of index a p-power. Then G is solvable.

Proposition

[Theorem 1] Let T be a nonabelian simple group with a subgroup $H < T$ satisfying $|T : H| = p^a$, for p a prime. Then one of the following holds:

(i) $T = A_n$ and $H = A_{n-1}$ with $n = p^a$;

(ii) $T = PSL(n, q)$, H is the stabilizer of a projective point or a hyperplane in $PG(n-1, q)$ and $|T : H| = (q^n - 1)/(q - 1) = p^a$;

(iii) $T = PSL(2, 11)$ and $H = A_5$;

(iv) $T = M_{11}$ and $H = M_{10}$;

(v) $T = M_{23}$ and $H = M_{22}$;

(vi) $T = PSU(4, 2)$ and H is a subgroup of index 27.

R.M. Guralnick, Subgroups of prime power index in a simple group, *J. Algebra*, **81**(1983), 304-311.

Normality of p-map

Lemma

Suppose $G = \langle r, \ell \rangle$ *such that* ℓ *is an involution,* $|r|$ *is even and* $|G : \langle r \rangle| = p^k$*, where* p *is an odd prime. Then* G *contains a normal Sylow* p*-subgroup.*

Lemma

Suppose $G = \langle t, r, \ell \rangle$ *such that* t, r, ℓ *are involutions,* $t\ell = \ell t$ *and* $\langle r, t \rangle \cong \mathbb{D}_{2n}$ *where* $|G : \langle r, t \rangle| = p^k$ *for an odd prime* p*. Then* $G/O_{2'}(G)$ *is isomorphic to either* S_4 *or a Sylow* 2*-group of* G*. Moreover, if* $p \geq 5$*, then* $G/O_{2'}(G) \ncong S_4$*.*

Proposition

Let G be a finite group with dihedral Sylow 2-subgroups. Let $O(G)$ denote the maximal normal subgroup of odd order. Then $G/O(G)$ is isomorphic to either a subgroup of $P\Gamma L(2, q)$ containing $PSL(2, q)$ where q is odd, or A_7, or a Sylow 2-subgroup of G.

Lemma

With the notations in last Lemma, suppose that $G/O_{2'}(G)$ *is isomorphic to a Sylow* 2*-subgroup of* G*, where* p *is odd. Then* G *contains a normal Sylow* p*-subgroup.*

Lemma

Suppose that $\mathcal{M}$ is an orientably-regular 3-map or a regular 3-map. Then either

(1) *$\mathcal{M}$ is normal; or*

(2) *$\mathcal{M}$ is nonorientable and nonnormal regular. Moreover, it is a regular covering of $\mathcal{C}(3,2)$, whose covering transformation group is a 3-group.*

C.H. Li and Jozef Širáň, Regular maps whose groups do not act faithfully on vertices, edges, or faces. *Euro. J. Combin.*, **26**(2005), 521-541.

Lemma

Suppose that $\mathcal{M}$ is an orientably-regular 2-map or a regular 2-map. Then either

(1) *$\mathcal{M}$ is normal; or*

(2) *$\mathcal{M}$ is a regular covering of one of the following maps $\overline{\mathcal{M}}$, whose covering transformation group is a 2-group:*

  (2.1) *$\overline{\mathcal{M}} = \mathcal{D}(m, e)$, where $m \geq 3$ is odd and $e^2 \equiv 1 \pmod{m}$ but $e \not\equiv 1 \pmod{m}$. $\mathcal{M}$ is nonnormal orientably-regular, more precisely, it is either a chiral map or reflexible and nonnormal regular;*

  (2.2) *$\overline{\mathcal{M}} = DM(m)$ and $\mathcal{M}$ is nonorientable and nonnormal regular;*

  (2.3) *$\overline{\mathcal{M}} = EM(m)$ and $\mathcal{M}$ is normal orientably-regular but nonnormal regular.*

C.H. Li and Jozef Širáň, Regular maps whose groups do not act faithfully on vertices, edges, or faces. *Euro. J. Combin.*, **26**(2005), 521-541.

Example

$O_3(G) = 1$: Let $G = S_4$. Take $r = (13)$, $t = (12)(34)$ and $\ell = (12)$. Then $\mathcal{M}(G; r, t, \ell)$ is a nonorientable nonnormal 3-map which has three vertices and six edges. Since $|\langle r, \ell\rangle| = 6$, the map has four faces. The genus g of the map is 1.

Example

$|O_3(G)| = 3$: Let $G = (\langle b\rangle \times \langle c\rangle) \rtimes \langle d, e\rangle$ with the defining relations

$$b^2 = c^2 = e^2 = d^9 = [b, c] = 1,$$

$$b^d = c, c^d = bc, d^e = d^{-1}, b^e = c, c^e = b.$$

Then $O_3(G) = \langle d^3\rangle$ and $G/O_3(G) \cong S_4$. Let $r = e, t = b, \ell = de$. Then $\mathcal{M}(G; r, t, \ell)$ is a nonorientable nonnormal 3-map which has 9 vertices, 18 edges and 4 faces. The genus g of the map is 7.

Example

$|O_3(G)| = 27$: Let $G = (\langle a\rangle \times \langle b\rangle \times \langle c\rangle) \rtimes \langle d, e, f\rangle$ with the defining relations

$$a^3 = b^3 = c^3, a^d = a^{-1}, b^d = b, c^d = c^{-1},$$

$$a^e = b^{-1}, b^e = a^{-1}, c^e = c^{-1}, a^f = ab^{-1}c^{-1}, b^f = a^{-1}bc^{-1}, c^f = a^{-1}b^{-1}$$

$$d^2 = e^2 = f^2 = 1, (de)^4 = 1, ef = fe, (df)^3 = 1, (def)^3 = 1.$$

Then $O_3(G) = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ and $G/O_3(G) \cong S_4$. Let $r = acd, t = e, \ell = f$. Then $\mathcal{M}(G; r, t, \ell)$ is a nonorientable nonnormal 3-map which has 27 vertices, 162 edges and 24 faces and the underlying graph is simple.

Example

Let $G = S_4 = \langle r, t, \ell \rangle$, where $r = (12)$, $t = (13)$ and $\ell = (13)(24)$ and $\mathcal{M} = \mathcal{M}(G; r, t, \ell)$. Since $\langle rt, t\ell \rangle = G$, we know $\mathcal{M}$ is a nonorientable and nonnormal regular 2-map, which has 4 vertices, 3 faces and 6 edges and the underlying graph is simple. Moreover, $O_2(G) \cong \mathbb{D}_4$ and $\overline{\mathcal{M}} = DM(6)$.

Let $G' \cong \mathbb{Z}_2 \times S_4 \cong \langle (56) \rangle \times \langle r, t, \ell \rangle = \langle r', t, \ell \rangle$, where $r' = (56)(12)$ and $\mathcal{M}' = \mathcal{M}'(G'; r', t, \ell)$. Clearly, we know $\mathcal{M}'$ is a nonorientable and nonnormal regular 2-map, which has 4 vertices, 6 faces and 12 edges and the underlying graph has doubled edges. Moreover, $O_2(G') \cong \mathbb{Z}_2 \times \mathbb{D}_4$ and $\overline{\mathcal{M}} = DM(6)$.

Example

Let $G = S_4 = \langle r, t, \ell \rangle$, where $r = (12)$, $t = (13)$ and $\ell = (24)$ and $\mathcal{M} = \mathcal{M}(G; r, t, \ell)$. Since $\langle rt, t\ell \rangle = A_4$, we know $\mathcal{M}$ is a nonnormal orientably-regular 2-map, which has 4 vertices, 4 faces and 6 edges and the underlying graph is simple. Moreover, $O_2(G) \cong \mathbb{D}_4$ and $\overline{\mathcal{M}} = EM(6)$.
Let $G' \cong \mathbb{Z}_2 \times S_4 \cong \langle (56) \rangle \times \langle r, t, \ell \rangle = \langle r', t, \ell \rangle$, where $r' = (56)(12)$ and $\mathcal{M}' = \mathcal{M}'(G'; r', t, \ell)$. Clearly, we know $\mathcal{M}'$ is a nonnormal orientably-regular 2-map, which has 4 vertices, 4 faces and 12 edges and the underlying graph has doubled edges. Moreover, $O_2(G') \cong \mathbb{Z}_2 \times \mathbb{D}_4$ and $\overline{\mathcal{M}'} = EM(6)$.

π-map

Theorem

Suppose that $\mathcal{M}$ is an orientably-regular π-map of odd order. Then the π-map $\mathcal{M}$ is solvable and normal.

Theorem

Suppose that $\mathcal{M}$ is a regular π-map of odd order. Then the regular π-map $\mathcal{M}$ is solvable if Aut$(\mathcal{M})$ has no composition factors isomorphic to $\mathrm{PSL}(2, q)$ for some odd prime power $q \neq 3$; and $\mathcal{M}$ is normal if and only if Aut$(\mathcal{M})$ has a normal Hall subgroup of odd order.

π-map

Theorem

Let $\mathcal{M}$ be a primitive orientably-regular π-map. Then π contains only one prime and $\mathcal{M}$ is solvable, moreover, $\mathcal{M}$ is normal if $2 \notin \pi$.

Theorem

Let $\mathcal{M}$ be a primitive regular π-map of odd order. Then

(1) *either π contains only one prime or π is the set of prime divisors of $\frac{q(q-1)}{2}$ or $\frac{q(q+1)}{2}$ where q is an odd prime power;*

(2) *$\mathcal{M}$ is solvable if and only if π contains only one prime, in that case, $\mathcal{M}$ is normal if the only prime in π is no less than 5.*

1.Shaofei Du, Yao Tian and Xiaogang Li, Orientably-Regular p-Maps and Regular p-Maps, journal of combinatorial theory, series A, 197 (2023), 105754.

2.Xiaogang Li, Yao Tian, On The Automorphism Groups of Regular Maps, Journal of Algebraic Combinatorics.

THANK YOU FOR YOUR ATTENTION!