Compatible Groups and Inverse Limits

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Motivation

Throughout, all groups and digraphs are finite

Let digraph Γ be G-arc-transitive and $v \in V\Gamma$.

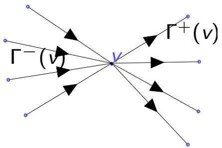

Figure: Local actions of G at v

$G_v^{\Gamma^{+}(v)}$: induced permutation group by G_v on $\Gamma^{+}(v)$.
$G_v^{\Gamma^{-}(v)}$: induced permutation group by G_v on $\Gamma^{-}(v)$.

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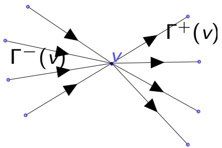

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$G_v^{\Gamma^-(v)}$: induced permutation group by G_v on $\Gamma^-(v)$.

Definition

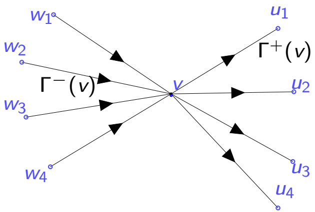
$G_v^{\Gamma^+(v)}$ and $G_v^{\Gamma^-(v)}$ are called *compatible* if they arise in this way.

Motivation

Question: Given two permutation groups L_1 and L_2, how to determine their compatibility?

Fact

- $G_v^{\Gamma^+(v)} \cong G_v^{[G_v:G_{vu_1}]}$ and $G_v^{\Gamma^-(v)} \cong G_v^{[G_v:G_{w_1v}]}$.
- $G_{vu_1} \cong G_{w_1v}$.



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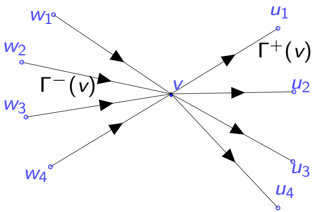

Theorem (Giudici et al. 2019)

L_1 and L_2 are compatible $\iff$ $\exists$ a group H with subgroups $K_1 \cong K_2$ s.t. $L_1 \cong H^{[H:K_1]}$ and $L_2 \cong H^{[H:K_2]}$.

Compatible groups

Definition

Two (abstract) groups L_1 and L_2 are *compatible* if $\exists$ G and $N_1 \cong N_2 \trianglelefteq G$ such that $L_\delta \cong G/N_\delta$ for $\delta = 1, 2$. Such a G is called a *witness*.

Example. C_2^2 and C_4 are compatible. We have a witness $C_2 \times C_4$ with isomorphic normal subgroups

$$(C_2 \times C_4)/(C_2 \times 1) \cong C_4$$

and

$$(C_2 \times C_4)/(1 \times C_2) \cong C_2^2.$$

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- 'Compatibility' $\implies$ '?'

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- **'?'** $\implies$ **'Compatibility'** (We will focus on this one in this talk)

Sufficient conditions of compatibility

If $\exists$ two normal series

$$1 = N_0 \trianglelefteq \cdots \trianglelefteq N_n = L_1$$

and

$$1 = M_0 \trianglelefteq \cdots \trianglelefteq M_n = L_2$$

such that $N_{i+1}/N_i \cong M_{i+1}/M_i$, we say L_1 and L_2 have compatible normal series.

Sufficient conditions of compatibility

Theorem (Length 2)

L_1 and L_2 have compatible normal series of length 2 $\implies$ compatible.

Proof

$\exists N_\delta \lhd L_\delta$ s.t. $N_1 \cong N_2$ and $L_1/N_1 \cong L_2/N_2$. Let $\sigma : L_1/N_1 \xrightarrow{\sim} L_2/N_2$, $\pi_\delta : L_\delta \to L_\delta/N_\delta$ canonical projection, and

$$G := \{(x, y) \in L_1 \times L_2 \mid \sigma \circ \pi_1(x) = \pi_2(y)\}.$$

Note that $N_1 \times 1, 1 \times N_2 \trianglelefteq G$. Also $G/(N_1 \times 1) \cong L_2$ and $G/(1 \times N_2) \cong L_1$. So G is a witness.

G is the inverse limit of the diagram:

$$\begin{array}{ccc} & & L_1 \\ & & \big\downarrow{\scriptstyle \sigma\circ\pi_1} \\ L_2 & \xrightarrow{\pi_2} & L_2/N_2 \end{array}$$

An example

S_4 and $C_2 \times A_4$ compatible.
$1 \lhd A_4 \lhd S_4$ and $1 \lhd 1 \times A_4 \lhd C_2 \times A_4$ are compatible normal series of length 2.

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$C_2 \times A_4$ and $\mathrm{SL}(2,3)$ compatible.
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S_4 and $\mathrm{SL}(2,3)$ NOT compatible.
They do not have any compatible subnormal series.

Quick introduction to inverse limits

Definition

Let I be a poset. An *inverse system* over I consists of:

(i) a group X_i, for each $i \in I$;

(ii) an homomorphism $f_{ij} : X_j \to X_i$, for every $i \leq j \in I$;

such that $f_{ii} = \mathrm{id}_{X_i}$ and $f_{ij} \circ f_{jk} = f_{ik}$ for $i \leq j \leq k \in I$.

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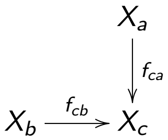
Example. $I = (\{a, b, c\}, \leq)$, $c \leq a$ and $c \leq b$.
An inverse system over I is as follows



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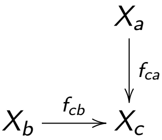
Let $((X_i), (f_{ij}))$ be an inverse system over I. Define

$$\varprojlim(X_i) := \{(x_i) \in \prod_{i \in I} X_i \mid \forall i \leq j \in I, f_{ij}(x_j) = x_i\}.$$

Example. $G := \{(x, y, z) \in X_a \times X_b \times X_c \mid f_{ca}(x) = f_{cb}(y) = z\}$ is the limit of



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Let $((X_i),(f_{ij}))$ an inverse system.

- Subsystem $((Y_i),(g_{ij}))$: an inverse system s.t. $Y_i \leq X_i$, $f_{ij}(Y_j) = Y_i$ & $g_{ij} := f_{ij}|_{Y_j}$.
- Normal subsystem: $\forall Y_i$, $Y_i \lhd X_i$.
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Proposition

- If (Y_i) is subsystem of (X_i), $\varprojlim(Y_i) \leq \varprojlim(X_i)$.
- If (Y_i) is normal subsystem of (X_i), $\varprojlim(Y_i) \trianglelefteq \varprojlim(X_i)$.
- $\varprojlim((X_i)/(Y_i)) \cong \varprojlim(X_i)/\varprojlim(Y_i)$.

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- $\varprojlim((X_i)/(Y_i)) \cong \varprojlim(X_i)/\varprojlim(Y_i)$. (Not generally true.)

Sufficient condition of compatibility: continued

Proposition (Length 3)

Let $L_1 := A._1 B._1 C$ and $L_2 := A._2 B._2 C$. If $\mathrm{Inn}(B._1 C)^B \leq (\mathrm{Aut}(B._2 C)_B)^B$ and $\mathrm{Inn}(B._2 C)^B \leq (\mathrm{Aut}(B._1 C)_B)^B$, L_1 and L_2 compatible.

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Sketch of proof.

Let $n := |C|$. Construct $G_{11} := A^n._1 B._1 C$, $G_{21} := A^n._2 B._1 C$, $G_{12} := A^n._1 B._2 C$, and $G_{22} := A^n._2 B._2 C$ s.t. $G_{11}/A^{n-1} \cong L_1$ and $G_{22}/A^{n-1} \cong L_2$.

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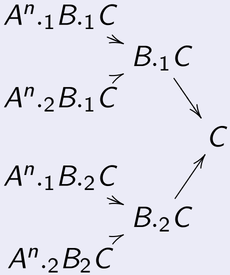

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Proposition (Current work)

Compatible normal series + extra hypothesis (\$) $\implies$ Compatible.

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Extra hypothesis ($)

Let $L_\delta := A_1 ._\delta A_2 ._\delta \cdots ._\delta A_\ell$ for $\delta = 1, 2$. For $2 \leq i \leq \ell - 1$,

$$\mathrm{Inn}(A_i ._1 \cdots ._1 A_\ell)^{A_i} \leq (\mathrm{Aut}(A_i ._2 \cdots ._2 A_\ell)_{A_i})^{A_i}$$

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Example. Compatible **central** series satisfy the extra hypothesis.
$(\mathrm{Inn}(A_i ._\delta \cdots ._\delta A_\ell)|_{A_i} = 1.)$

Sufficient condition of compatibility: continued

Corollary

(i) *All nilpotent groups of the same order are compatible to each other.*

(ii) *All groups of the same square-free order are compatible to each other.*

Further comments

1. Can we remove the extra hypothesis in previous proposition?

Conjecture

Compatible normal series $\Longrightarrow$ Compatible.

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Lemma (Sims)

Compatible $\Longrightarrow$ Compatible subnormal series.

Conjecture

Compatible $\Longrightarrow$ Compatible normal series.

Proposition (Progress so far)

L_1 and L_2 compatible + their composition factors are all non-abelian $\Longrightarrow$ L_1 and L_2 have compatible normal series.

Further comments

3. Can we prove A_4 and C_{12} incompatible?

Proposition (Progress so far)

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4. Some other applications of inverse limits? e.g. Subdirect subgroups.

Thereom (Goursat's Lemma)

The subdirect subgroup of $G \times H$ is the inverse limit of

$$\begin{array}{ccc} G & & \\ & \searrow^{p} & \\ & & C \\ & \nearrow_{q} & \\ H & & \end{array}$$

for some group C and surjective p, q.