Recent Advances in Graphical Regular Representations of Groups

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König's conjecture

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As to graphs, the problem of whether a group can be represented as the automorphism group of a graph was considered at a very early stage of graph theory.

König conjectured in his 1936 book "Theorie der endlichen und unendlichen Graphen', the first textbook on the field of graph theory, that every finite group is the automorphism group of a finite graph.

Frucht's theorem

- König's conjecture was proved by Frucht in 1939[1].

[1] R. Frucht, Herstellung von Graphen mit vorgegebener abstrakter Gruppe, *Compositio Math.*, 6 (1939), 239–250.

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- In 1949, Frucht[2] proved a stronger version stating that every finite group is the automorphism group of a cubic graph.

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In the above results, the group may not act transitively on the vertex set and may not have the same order as the graph.

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[4] C. D. Godsil, GRRs for nonsolvable groups, *Algebraic methods in graph theory, Vol. I, II (Szeged, 1978)*, pp. 221–239, Colloq. Math. Soc. János Bolyai, 25, North-Holland, Amsterdam-New York, 1981.

Groups admitting GRRs

Except the following solvable groups, every finite group admits a GRR.

- abelian groups of exponent greater than 2,
- generalized dicyclic groups (nonabelian groups R that have an abelian normal subgroup A of index 2 and an element $x \in R \setminus A$ of order 4 such that $x^{-1}ax = a^{-1}$ for all $a \in A$),
- Z_2^2, Z_2^3, Z_2^4, D_6, D_8, D_{10}, $Q_8 \times Z_3$, $Q_8 \times Z_4$, A_4,
- $\langle a, b \mid a^8 = b^2 = 1,\ bab = a^5 \rangle$,
- $\langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ abc = bca = cab \rangle$,
- $\langle a, b, c \mid a^3 = b^3 = c^2 = 1,\ ab = ba,\ (ac)^2 = (bc)^2 = 1 \rangle$,
- $\langle a, b, c \mid a^3 = b^3 = c^3 = 1,\ ac = ca,\ bc = cb,\ c = a^{-1}b^{-1}ab \rangle$.

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- What about (existence/proportion of) GRRs of prescribed valency?
- What about variants (e.g. the digraph version) of GRR?

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Theorem (Babai 1980 *Period. Math. Hungar.*)

Except Z_2^2, Z_2^3, Z_2^4, Z_3^2 and Q_8, every finite group admits a DRR.

Enumerating DRRs

Babai[8] and Godsil[9] conjectured that the proportion of subsets S of a group R such that $\mathrm{Cay}(R, S)$ is a DRR approaches 1 as $|R| \to \infty$.

[8] L. Babai and C. D. Godsil, On the automorphism groups of almost all Cayley graphs, *European J. Combin.*, 3 (1982), 9–15.

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Theorem (Morris-Spiga 2021 *Israel J. Math.*)

- For a group R of order r, when r is sufficiently large, the proportion of subsets S of R such that $\mathrm{Cay}(R, S)$ is a DRR of R is at least $1 - 2^{-\frac{br^{0.499}}{4\log_2^3 r}+2}$, where b is an absolute constant.
- Let $\mathrm{CD}(R)$ denote the set of Cayley digraphs on R up to isomorphism and let $\mathrm{DRR}(R)$ denote the set of DRRs of R up to isomorphism. Then $|\mathrm{DRR}(R)|/|\mathrm{CD}(R)| \to 1$ as $|R| \to \infty$.

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Enumerating GRRs

A similar conjecture of Babai-Godsil-Imrich-Lovász states[8] that the proportion of inverse-closed subsets S of a group R with GRR such that $\mathrm{Cay}(R, S)$ is a GRR approaches 1 as $|R| \to \infty$.

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Theorem (X.-Zheng 2023 *PLMS*)

- For a group R of order r with GRR, when r is sufficiently large, the proportion of inverse-closed subsets S of R such that $\mathrm{Cay}(R,S)$ is a GRR of R is at least $1 - 2^{-\frac{r^{0.499}}{8\log_2^3 r}+\log_2^2 r+3}$.
- Let $\mathrm{CG}(R)$ denote the set of Cayley graphs on R up to isomorphism and let $\mathrm{GRR}(R)$ denote the set of GRRs of R up to isomorphism. For a group R of order r with GRR, when r is sufficiently large, $|\mathrm{GRR}(R)|/|\mathrm{CG}(R)| \geq 1 - 2^{-\frac{r^{0.499}}{8\log_2^3 r}+\log_2^2 r+3}$.

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Automorphism groups of Cayley graphs

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Conder and Poznanović[10] showed that for every integer $k \geqslant 3$, there exists infinitely many k-valent GRRs of some groups.

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Godsil[12] proved that every alternating group A_n with $n \geqslant 5$ has a cubic GRR.

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The Fang-Li-Wang-Xu conjecture

The special interest in cubic GRRs of finite nonabelian simple groups is partially because of

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In 2002, Fang, Li, Wang and Xu[13] conjectured that every finite nonabelian simple group has a cubic GRR.

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In 2002, Fang, Li, Wang and Xu[13] conjectured that every finite nonabelian simple group has a cubic GRR.

They also verified their conjecture for some small groups and the infinite family of Suzuki groups ${}^2\mathrm{B}_2(q)$, where $q = 2^{2c+1} \geqslant 8$.

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- There exist involutions x and y in G such that the proportion of involutions z for $\mathrm{Cay}(G, \{x, y, z\})$ to be a cubic GRR of G approaches 1 as $q \to \infty$.

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Since $\mathrm{PSL}_2(7)$ is a counterexample, the Fang-Li-Wang-Xu conjecture should be modified to: There are only finitely many finite nonabelian simple groups without cubic GRR.

Spiga's conjectures

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(i) Except for a finite number of cases and for the groups $\mathrm{PSL}_2(q)$, every finite nonabelian simple group G contains an element x and an involution y such that $\mathrm{Cay}(G,\{x,x^{-1},y\})$ is a cubic GRR of G.

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(ii) Except for a finite number of cases, every finite nonabelian simple group G contains three involutions x, y and z such that $\mathrm{Cay}(G, \{x, y, z\})$ is a cubic GRR of G.

(iii) The proportion of cubic Cayley graphs (up to isomorphism) on a finite nonabelian simple group G that are GRRs approaches 1 as $|G| \to \infty$.

Lie type of large rank

Theorem (X. 2020 *JCTB*)

Let G be a finite simple group of Lie type of rank at least 9. Then there exists an element x of prime order in G such that the proportion of involutions y in G for $\mathrm{Cay}(G, \{x, x^{-1}, y\})$ to be a cubic GRR of G approaches 1 as $|G| \to \infty$.

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- The theorem gives an affirmative answer to Spiga's conjecture (i) for finite simple groups of Lie type of rank at least 9, and also gives evidence for Spiga's conjecture (iii).

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- The theorem gives an affirmative answer to Spiga's conjecture (i) for finite simple groups of Lie type of rank at least 9, and also gives evidence for Spiga's conjecture (iii).
- The theorem reduces the verification of modified Fang-Li-Wang-Xu's conjecture to finite simple groups of Lie type of rank at most 8.

Spiga's conjecture (i)

Leemans and Liebeck[14] proved that for every finite nonabelian simple group G except A_7, $\mathrm{PSL}_2(q)$, $\mathrm{PSL}_3(q)$ and $\mathrm{PSU}_3(q)$, there exists a pair of generators x and y of G with y an involution and $\mathrm{Aut}(G, \{x, x^{-1}, y\}) = 1$.

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The groups $\mathrm{PSL}_3(q)$ and $\mathrm{PSU}_3(q)$ are genuine exceptions of the above conclusion and hence counterexamples to Spiga's conjecture (i).

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The groups $\mathrm{PSL}_3(q)$ and $\mathrm{PSU}_3(q)$ are genuine exceptions of the above conclusion and hence counterexamples to Spiga's conjecture (i). But they are the only counterexamples because:

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Theorem (X-Zheng-Zhou 2022 *JA*)

Except for a finite number of cases and for the groups $\mathrm{PSL}_2(q)$, $\mathrm{PSL}_3(q)$ and $\mathrm{PSU}_3(3)$, every finite nonabelian simple group G contains an element x and an involution y such that $\mathrm{Cay}(G, \{x, x^{-1}, y\})$ is a cubic GRR of G.

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Combining a few of the above theorems, we now know that the modified Fang-Li-Wang-Xu's conjecture is true, i.e., there are only finitely many finite nonabelian simple groups without cubic GRR.

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- $\mathrm{PSL}_2(7)$ and $\mathrm{PSU}_3(3)$ are examples without cubic GRR.
- Spiga's conjectures (ii) and (iii) are still open.

k-valent GRRs of nonabelian simple groups

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Theorem (X. 2023+)

- For each integer $k \geq 5$, there is a k-valent GRR of A_n for all $n \geq \max\{14, 6\lceil k/2 \rceil - 12\}$.

k-valent GRRs of nonabelian simple groups

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- For each integer $k \geq 5$, there is a k-valent GRR of A_n for all $n \geq \max\{14, 6\lceil k/2 \rceil - 12\}$.
- For each integer $k \geq 5$, there exists a constant $N(k)$ such that every finite simple group of Lie type of rank at least $N(k)$ has a k-valent GRR.

Variants

Variants

- oriented regular representation (ORR)
- ORR of nonabelian simple groups of out-valency 2
- graphical Frobenius representation (GFR)
- DmRR
- GmRR

Thank you for listening!