The Product of a Finite Group And a Cyclic Group

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Factorizations of groups

Definition (Factorizations of groups)

- *A group X is said to be properly factorizable if $X = GC$ for two proper subgroups G and C of X, while the expression $X = GC$ is called a factorization of X, and X is the product group of G and C.*
- *We say that X has an exact factorization if $G \cap C = 1$.*

Factorizations of groups

Factorizations of groups naturally arise from the well-known Frattini's argument, including its version in permutation groups.

Proposition (Frattini's argument)

Let X be a group acting transitively on a set Ω, G be a subgroup of X and X_α be a point stabilizer of X. If G acts transitively on Ω, then $X = GX_\alpha$.

Proposition (Lucchini)

If X is a transitive permutation group of degree n with a cyclic point-stabilizer, then $|X| \leq n(n-1)$.

G acts regularly on Ω if $G \cap X_\alpha = 1$. A group G is said a Burnside group if every permutation group containing a regular subgroup isomorphic to G is either 2-transitive or imprimitive.

Factorizations of groups

Proposition (Burnside)

Every cyclic group of order of p^m $(m > 1)$ is a Brunside group.

Proposition (Schur)

Every cyclic group of composite order is a Brunside group.

Proposition (H. Wielandt)

Every dihedral group is a Burnside group.

Proposition (Scott)

Every generalized quaternion group is a Burnside group.

Factorizations of groups

Proposition (Itô)

If X has a factorization $X = GC$ where both G and C are abelian subgroups of X, Then X is metabelian, that is X' is abelian.

Proposition (Wielandt and Kegel)

The product of two nilpotent subgroups must be soluble.

Proposition (Douglas)

The product of two cyclic groups must be super-solvable.

Proposition (V. S. Monakhov)

The finite group $X = GC$ is solvable, where both G and C are subgroups with cyclic subgroups of index no more than 2.

Factorizations of groups

The factorizations of the finite almost simple groups were determined in M. W. Licheck, C. E. Prager and J. Saxl. The factorizations of almost simple groups with a solvable factor were determined in C.H. Li and B. Z. Xia.

Factorizations of groups

Set $Q = \langle a, b \mid a^{2n} = 1, b^2 = a^n, a^b = a^{-1} \rangle \cong Q_{4n}$, $D = \langle a, b \mid a^n = b^2 = 1, a^b = a^{-1} \rangle \cong D_{2n}$ and $C = \langle c \mid c^m = 1 \rangle$ where $n \geq 2$. Let $G \in \{Q, D\}$. Then we have the following results.

Theorem

Suppose that $X = X(G)$ has an exact factorization $X = GC$. Let M be the subgroup of the biggest order in X such that $\langle c \rangle \leq M \subsetneqq \langle a \rangle \langle c \rangle$. Then one of items in the following table holds.

Table: The forms of M, M_X and X/M_X

Case	M	M_X	X/M_X
1	$\langle a \rangle \langle c \rangle$	$\langle a \rangle \langle c \rangle$	$\mathbb{Z}_2$
2	$\langle a^2 \rangle \langle c \rangle$	$\langle a^2 \rangle \langle c^2 \rangle$	D_8
3	$\langle a^2 \rangle \langle c \rangle$	$\langle a^2 \rangle \langle c^3 \rangle$	A_4
4	$\langle a^3 \rangle \langle c \rangle$	$\langle a^3 \rangle \langle c^4 \rangle$	S_4
5	$\langle a^4 \rangle \langle c \rangle$	$\langle a^4 \rangle \langle c^3 \rangle$	S_4

Factorizations of groups

Proposition

Let H be a subgroup of G. Then $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}\,(H)$.

Using above theorem and proposition, we can get the following theorem:

Theorem

Let $G \in \{Q, D\}$ and $X = X(G)$, and M be defined as above. Then we have $\langle a^2, c\rangle \leq C_X(\langle c\rangle_X)$ and $|X : C_X(\langle c\rangle_X)| \leq 4$. Moreover, if $\langle c\rangle_X = 1$, then $M_X \cap \langle a^2\rangle \lhd M_X$.

Skew product groups

- X is called a skew product group of G if X has an exact factorization $X = GC$ where C is a cyclic group and is core-free.
- Let $A = G.\langle t \rangle$, where $G \lhd A$, be a group and $t^l = g \in G$. Then t induces an automorphism τ of G by conjugacy. Recall that by the cyclic extension theory of groups, this extension is valid if and only if

$$\tau^l = \mathrm{Inn}(g) \quad \text{and} \quad \tau(g) = g.$$

Skew product groups

Using Tabel 1 and Theorem 1.14, we get

Table: The forms of M, M_X and X/M_X

Case	M	M_X	X/M_X
1	$\langle a\rangle\langle c\rangle$	$(\langle a^2\rangle \rtimes \langle c\rangle).\langle a\rangle$	$\mathbb{Z}_2$
2	$\langle a^2\rangle\langle c\rangle$	$\langle a^2\rangle \rtimes \langle c^2\rangle$	D_8
3	$\langle a^2\rangle\langle c\rangle$	$\langle a^2\rangle \rtimes \langle c^3\rangle$	A_4
4	$\langle a^3\rangle\langle c\rangle$	$(\langle a^6\rangle\langle c^4\rangle).\langle a^3\rangle$	S_4
5	$\langle a^4\rangle\langle c\rangle$	$\langle a^4\rangle \rtimes \langle c^3\rangle$	S_4

$G = Q$ and $M = \langle a\rangle\langle c\rangle$

Suppose that $X = X(Q)$, $M = \langle a\rangle\langle c\rangle$ and $\langle c\rangle_X = 1$. Set $R := \{a^{2n} = c^m = 1,\, b^2 = a^n,\, a^b = a^{-1}\}$. Then

$$\begin{aligned} X &= ((\langle a^2\rangle \rtimes \langle c\rangle).\langle a\rangle).\langle b\rangle \\ &= \langle a, b, c | R, (a^2)^c = a^{2r}, c^a = a^{2s}c^t, c^b = a^u c^v\rangle, \end{aligned}$$

where $r^{t-1} \equiv r^{v-1} \equiv 1 (\mathrm{mod}\ n)$, $t^2 \equiv 1 (\mathrm{mod}\ m)$, $2s\sum_{l=1}^{t} r^l + 2sr \equiv 2sr + 2s\sum_{l=1}^{v} r^l - u\sum_{l=1}^{t} r^l + ur \equiv 2(1-r) (\mathrm{mod}\ 2n)$, $2s\sum_{l=1}^{w} r^l \equiv u\sum_{l=1}^{w}(1-s(\sum_{l=1}^{t} r^l + r))^l \equiv 0 (\mathrm{mod}\ 2n) \Leftrightarrow w \equiv 0 (\mathrm{mod}\ m)$, and moreover, if $2 \mid n$, then $u(\sum_{l=0}^{v-1} r^l - 1) \equiv 0 (\mathrm{mod}\ 2n)$ and $v^2 \equiv 1 (\mathrm{mod}\ m)$; if $2 \nmid n$, then $u\sum_{l=1}^{v} r^l - ur \equiv 2sr + (n-1)(1-r) (\mathrm{mod}\ 2n)$ and $v^2 \equiv t (\mathrm{mod}\ m)$; and if $t \neq 1$, then u is even.

$G = Q$ and $M = \langle a^2\rangle\langle c\rangle$

Suppose that $X = X(Q)$, $M = \langle a^2\rangle\langle c\rangle$, $X/M_X \cong D_8$ and $\langle c\rangle_X = 1$. Set $R := \{a^{2n} = c^m = 1, b^2 = a^n, a^b = a^{-1}\}$. Then

$$\begin{aligned} X &= ((((\langle a^2\rangle \rtimes \langle c^2\rangle).\langle a\rangle).\langle b\rangle).\langle c\rangle \\ &= \langle a, b, c | R, (a^2)^{c^2} = a^{2r}, (c^2)^a = a^{2s}c^{2t}, \\ &\quad (c^2)^b = a^{2u}c^2, a^c = bc^{2w}\rangle, \end{aligned}$$

where either $w = 0$ and $r = s = t = u = 1$; or
$w \neq 0$, $s = u^2 \sum_{l=0}^{w-1} r^l + \frac{un}{2}$, $t = 2wu + 1$,
$r^{2w} - 1 \equiv (u \sum_{l=1}^{w} r^l + \frac{n}{2})^2 - r \equiv 0 (\mathrm{mod}\ n)$,
$s \sum_{l=1}^{t} r^l + sr \equiv 2sr - u \sum_{l=1}^{t} r^l + ur \equiv 1 - r (\mathrm{mod}\ n)$,
$2w(1 + uw) \equiv nw \equiv 2w(r - 1) \equiv 0 (\mathrm{mod}\ \frac{m}{2})$ and
$2^{\frac{1+(-1)^u}{2}} \sum_{l=1}^{i} r^l \equiv 0 (\mathrm{mod}\ n) \Leftrightarrow i \equiv 0 (\mathrm{mod}\ \frac{m}{2})$.

$G = Q$ and $M = \langle a^2\rangle\langle c\rangle$

Suppose that $X = X(Q)$, $M = \langle a^2\rangle\langle c\rangle$, $X/M_X \cong A_4$ and $\langle c\rangle_X = 1$. Set $R := \{a^{2n} = c^m = 1,\ b^2 = a^n,\ a^b = a^{-1}\}$. Then

$$\begin{aligned} X &= ((((\langle a^2\rangle \rtimes \langle c^3\rangle).\langle a\rangle).\langle b\rangle).\langle c\rangle \\ &= \langle a, b, c | R, (a^2)^c = a^{2r}, (c^3)^a = a^{2s}c^3, (c^3)^b = a^{2u}c^3, \\ &\quad a^c = bc^{\frac{im}{2}}, b^c = a^x b\rangle, \end{aligned}$$

where $n \equiv 2 (\mathrm{mod}\ 4)$ and either

(1) $i = s = u = 0$, $r = x = 1$; or

(2) $i = 1$, $6 \mid m$, $r^{\frac{m}{2}} \equiv -1 (\mathrm{mod}\ n)$ with $\mathrm{o}(r) = m$, $s \equiv \frac{r^{-3}-1}{2} + \frac{n}{2} (\mathrm{mod}\ n)$, $u \equiv \frac{r^3-1}{2r^2} + \frac{n}{2} (\mathrm{mod}\ n)$, $x \equiv -r + r^2 + \frac{n}{2} (\mathrm{mod}\ n)$.

Suppose that $X = X(Q)$, $M = \langle a^3\rangle\langle c\rangle$, $X/M_X \cong S_4$ and $\langle c\rangle_X = 1$.

Theorem

$\langle a^3\rangle \lhd X$.

Set $R := \{a^{2n} = c^m = 1,\ b^2 = a^n,\ a^b = a^{-1}\}$. Then

$$\begin{aligned} X &= (((\langle a^3\rangle \rtimes \langle c^2\rangle).\langle b\rangle).\langle a\rangle).\langle c\rangle \\ &= \langle a, b, c | R, a^{c^4} = a^r, b^{c^4} = a^{1-r}b, (a^3)^{c^{\frac{m}{4}}} = a^{-3}, a^{c^{\frac{m}{4}}} = bc^{\frac{3m}{4}}\rangle, \end{aligned}$$

where $m \equiv 4 \pmod 8$ and r is of order $\frac{m}{4}$ in $\mathbb{Z}_{2n}^*$.

$G = Q$ and $M = \langle a^4\rangle\langle c\rangle$

Suppose that $X = X(Q)$, $M = \langle a^4\rangle\langle c\rangle$, $X/M_X \cong S_4$ and $\langle c\rangle_X = 1$. Set $R := \{a^{2n} = c^m = 1,\ b^2 = a^n,\ a^b = a^{-1}\}$. Then

$$\begin{aligned} X &= (((\langle a^2, b\rangle\langle c^3\rangle).\langle c\rangle).\langle a\rangle \\ &= \langle a, b, c | R, (a^4)^c = a^{4r}, (c^3)^{a^2} = a^{4s}c^3, \\ &\qquad (c^3)^b = a^{4u}c^3, (a^2)^c = bc^{\frac{im}{2}}, b^c = a^{2x}b, c^a = a^{2(1+2z)}c^{1+\frac{jm}{3}}\rangle, \end{aligned}$$

where either

(1) $i = 0$, $r = j = 1, x = 3, s = u = z = 0$; or

(2) $i = 1$, $n \equiv 4 \pmod 8$, $6 \mid m$, $r^{\frac{m}{2}} \equiv -1 \pmod{\frac{n}{2}}$, $o(r) = m$,
$s \equiv \frac{r^{-3}-1}{2} + \frac{n}{4} \pmod{\frac{n}{2}}$,
$u \equiv \frac{r^3-1}{2r^2} + \frac{n}{4} \pmod{\frac{n}{2}}$, $x \equiv -r + r^2 + \frac{n}{4} \pmod{\frac{n}{2}}$,
$1 + 2z \equiv \frac{1-r}{2r} \pmod{\frac{n}{2}}$, $j \in \{1, 2\}$.

$G = D$

Theorem

Let $G = D$ *and* $X = X(D) = G\langle c\rangle$*, where* $m = o(c) \geq 2$*,* $G \cap \langle c\rangle = 1$ *and* $\langle c\rangle_X = 1$*. Set* $R := \{a^n = b^2 = c^m = 1,\ a^b = a^{-1}\}$*. Then one of following holds:*

1. $X = \langle a, b, c | R, (a^2)^c = a^{2r}, c^a = a^{2s}c^t, c^b = a^u c^v\rangle$,
2. $X = \langle a, b, c | R, (a^2)^{c^2} = a^{2r}, (c^2)^b = a^{2s}c^2, (c^2)^a = a^{2u}c^{2v}, a^c = bc^{2w}\rangle$,
3. $X = \langle a, b, c | R, a^{c^3} = a^r, (c^3)^b = a^{2u}c^3, a^c = bc^{\frac{im}{2}}, b^c = a^x b\rangle$,
4. $X = \langle a, b, c | R, (a^2)^{c^3} = a^{2r}, (c^3)^b = a^{\frac{2(l^3-1)}{l^2}}c^3, (a^2)^c = bc^{\frac{im}{2}}, b^c = a^{2(-l+l^2+\frac{n}{4})}b, c^a = a^{2+4z}c^{2+3d}\rangle$,
5. $X = \langle a, b, c | R, a^{c^4} = a^r, b^{c^4} = a^{1-r}b, (a^3)^{c^{\frac{m}{4}}} = a^{-3}, a^{c^{\frac{m}{4}}} = bc^{\frac{3m}{4}}\rangle$,

where the above parameters meet certain conditions.

G is a p-group.

Theorem

Let $X = GC$ be a group, where G is a p-group and C is a cyclic group such that $G \cap C = 1$. Set $C = C_1 \times C_2$, where C_1 is the Sylow p-subgroup of C. If $C_X = 1$, then $F(X) = O_p(X) = G_1C_1$, where $G_1 = O_p(X) \cap G \neq 1$ and $G_1C_1 \rtimes C_2 \lhd X$.

G is an abelian p-group.

Theorem

$F(X)$ is the Sylow p-subgroup of X.

Theorem

If $X = \langle g, \sigma \rangle$ where $g \in G \cong \mathbb{Z}_p^n$ and $C = \langle \sigma \rangle$, then $X \leq \mathrm{AGL}(n, p)$.

G is a maximal class 2-group.

Theorem

Let $X = GC$ be a group, where C is a cyclic group, and suppose that G is a maximal class 2-group and $|G| = 2^n \geq 32$. Assume that $G \cap C = 1$ and that $C_X = 1$. Then X is a 2-group.

Theorem

Let $X = GC$ be a 2-group, where G is a maximal class group, C is a cyclic group and $G \cap C = 1$. If $C_X = 1$, then G_X is $\langle a_0 \rangle$, $\langle a^2, b \rangle$ or G.

G is a maximal class 2-group.

Theorem

Let $X = GC$ be a 2-group, where G is a maximal class group, C is a cyclic group and $G \cap C = 1$. Set R is the defined relation of G. Then X is isomorphic to one of the following groups:

(1) $X = \langle a, b, c | R, a^c = a^r, b^c = a^s b \rangle$, *where* $r^{2^m} \equiv 1(2^{n-1})$, *and* $r^{2^{m-1}} \not\equiv 1(2^{n-1})$ *or* $s\frac{r^{2^{m-1}}-1}{r-1} \not\equiv 0(2^{n-1})$. *Moreover, if G is a semidihedral 2-groups, then* $2|s$;

(2) $X = \langle a, b, c | R, (a^2)^{c^2} = a^2, (c^2)^a = a^{2s}c^{-2}, (c^2)^b = a^{2u}c^2, a^c = bc^{2y} \rangle$, *where* $sy \equiv 1 + i2^{n-3} (\text{mod } 2^{n-2})$ *and* $yu \equiv -1 (\text{mod } 2^{n-3})$, $i = 1$ *if G is a generalized quaternion group and $i = 0$ if G is either a dihedral group or a semidihedral group.*

Continue

(3) $X = \langle a, b, c | R, (a^2)^c = a^{2r},\ c^b = a^{2s}c,\ c^a = a^{2t}b^u c^v \rangle$, where $r^{2^m} \equiv 1 (\text{mod } 2^{n-2}),\ s\sum_{l=1}^{2^m} r^l \equiv 0 (\text{mod } 2^{n-2})$, either

(3.1) $u = 0,\ r^{v-1} \equiv 1 (\text{mod } 2^{n-2})$, $(s + 2t)r \equiv (1 - r) + s\sum_{l=1}^{v} r^l (\text{mod } 2^{n-2}),\ t\sum_{l=1}^{2^m} r^l \equiv 0 (\text{mod } 2^{n-2}), v^2 \equiv 1 (\text{mod } 2^m)$ and $1 - r \equiv tr + t\sum_{l=1}^{v} r^l (\text{mod } 2^{n-2})$; or

(3.2) $u = 1,\ r^{v-1} + 1 \equiv 0 (\text{mod } 2^{n-2}), (sr + 1 - r)\sum_{l=0}^{v-1} r^l \equiv (s + 2t + 1)r (\text{mod } 2^{n-1}), (t(1 - r^{-1}) + s\sum_{l=0}^{v-1} r^l)\sum_{l=0}^{2^{m-1}-1} r^{2l} \equiv 0 (\text{mod } 2^{n-2}), r^2[t(1 - r^{-1}) + s\frac{r^v - 1}{r - 1}]\frac{r^{v-1} - 1}{r^2 - 1} + 2^{n-3}i \equiv 0 (2^{n-2})$.

Thanks!