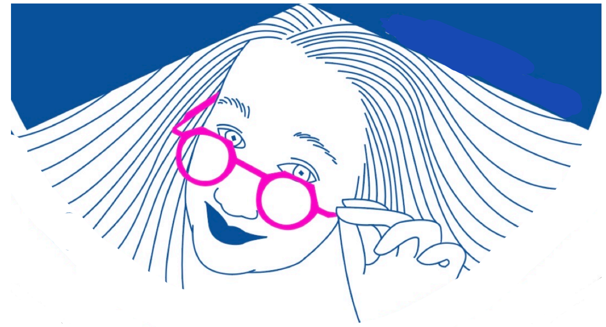
Imperial College London



The orbital diameter of primitive permutation groups

Kamilla Rekvényi

Group Theory Seminar

15/05/2023

$G \leq \mathrm{Sym}(\Omega)$, finite

transitive

$\forall a, b \in \Omega \ \exists g \in G$ s.t. $a^g = b$

$\exists!$ orbit, $a^G = \Omega$.

e.g. $G = S_n$

$G = \langle (1 \dots n) \rangle$

primitive

A transitive group G acts primitively if it preserves no non-trivial partition of Ω

e.g. $G = S_n$

$G = A_n$

$G = \langle (1 \dots p) \rangle$

non e.g. $G = \langle (1 \dots 2k) \rangle$ for some k fixed

135.... 124....

O'Nan - Scott Thm

Classifies the prim. perm. grps to be one of the five types

- Affine
- Almost simple
- Simple diagonal action
- product actions
- twisted wreath actions

$G \leq \mathrm{Sym}(\Omega)$ transitive & finite

G acts componentwise on $\Omega \times \Omega$

$\boxed{\text{orbital}}$ orbit of G on $\Omega \times \Omega$

$\Delta = \{(a,a) \mid a \in \Omega\}$

diagonal orbital

non-diagonal orbital

$(a,b)^G$, $a \neq b$

$\boxed{\text{orbital graph}}$

Γ - non-diagonal orbital

$\boxed{\text{vertices}}$

Ω

$\boxed{\text{edges}}$

$a - b \iff \{a,b\} \in \Gamma$

$\boxed{\text{Theorem (Higman)}}$

All non-diagonal orbital graphs of a group G are connected

$\iff$

G acts primitively on Ω.

$\boxed{\text{orbital diameter}}$ (orbdiam(G))

the supremum of the diameters of all orbital graphs

Example 1

G 2-transitive on Ω

Q: What is orbdiam (G)?

Example 1

G 2-transitive on Ω

$$\Downarrow$$

$\exists$ unique non-diagonal orbital

$$\Downarrow$$

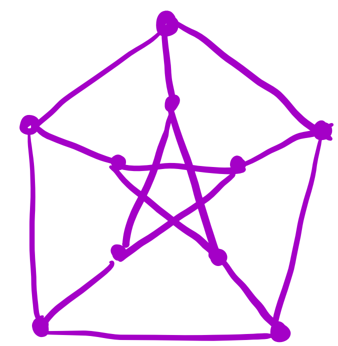
$$\text{orbdiam}(G) = 1$$

Example 2

$G = S_5$, $I = \{1,2,3,4,5\}$, $\Omega = \left\{ \begin{array}{l} \text{size 2} \\ \text{subsets of } I \end{array} \right\}$

orbitals

- Δ
- $\Gamma_1 = \{ (A,B) \mid |A \cap B| = 0 \}$
- $\Gamma_2 = \{ (A,B) \mid |A \cap B| = 1 \}$



Petersen graph $\Downarrow$ diam = 2

complement of Petersen graph $\Downarrow$ diam = 2

alternatively

$\text{orbdiam}(G) \leq \underbrace{\text{rank}(G)}_{\#\text{ orbitals}} - 1 = 2$ in this example

Theorem (Liebeck, MacPherson, Tent)

Classified ∞ families $\mathcal{C}$ of groups such that $\exists t \in \mathbb{N}$, $\forall G \in \mathcal{C}$, $\mathrm{orbdiam}(G) \leq t$.

GOALS

G1 find explicit bounds

G2 classify groups with small orbdiams

$\boxed{\text{Diagonal type}}$

T - simple , $T^k = \overbrace{T \times \cdots \times T}^{k}$

$D = \{(a, \ldots, a) \mid a \in T\} \cong T$

$\Omega = (T^k : D)$

$T^k \trianglelefteq G \leq N_{Sym(\Omega)}(T^k) \cong T^k . (Out(T) \times S_k)$

$$\mathrm{orbdiam}(G) \;\overset{\text{connection}}{\sim}\; \underbrace{c(T)}_{\substack{\text{conjugacy}\\ \text{width}}} \;\le\; \underbrace{cn(T)}_{\substack{\text{covering}\\ \text{number}}}$$

$$C = \{t^{\pm a} \mid a \in T\} \qquad 1 \cup C \cup \dots \cup C^{c(T)} = T \qquad C^{cn(T)} = T$$

<u>Thm</u> (R. 22)

$$\mathrm{orbdiam}(T \times T) = c(T)$$

Pf <u>Sketch</u> ("$\leq$")

$$G = T \times T \quad \& \quad D = \{(a,a) \mid a \in T\}$$

non-diag orbitals: $\{D, D(1,t)\}^G \qquad t \in T \setminus 1$

$$D \text{ — } D(1,t) \qquad \times (1, t^{-1})$$

$$D \text{ — } D(1,t^{-1}) \qquad \times (a,a)$$

$$D \text{ — } D(1, t^{\pm 1})(a,a) = D(a^{-1}a, a^{-1}t^{\pm 1}a)$$

$$D \text{ — } D(1, t^{\pm a}) \text{ — } D(1, t^{\pm b} t^{\pm a}) \ldots$$

$\forall h \in T, \quad h = t^{\pm a_1} \ldots t^{\pm a_c}$

where $c \leq c(T)$ $\blacksquare$

$\boxed{\text{Theorem (R '22)}}$ G1

$$X \underset{\text{primitive}}{\leq} S_k$$

1. $\frac{1}{2}(k-1)\, c(T) \leq \operatorname{orbdiam}(T^\ell X)$

2. $\operatorname{orbdiam}(T^\ell S_k) \leq 24(k-1)\, c^2(T)$

3. $\operatorname{orbdiam}(T^2) = c(T)$

$\boxed{\text{Conjugacy width}}$

$\boxed{\text{Theorem}}$ T – simple group of Lie type

r – Lie rank of T

R'22

$r - 3 \leq c(T) \leq cn(T) \leq dr$ for some $d \in \mathbb{N}$.

Ellers Gordeev Herzog

Arad, Chillag, Moran $cn(T) = 2 \Leftrightarrow T = J_1$

R'22 $c(T) = 3 \Leftrightarrow cn(T) = 3 \Leftrightarrow T$ is one of

6 ∞-te families

13 sporadic groups

or A_7

e.g. $PSL_2(q)$

Monster

$\boxed{\text{Theorem (R'22)}}$ G2

Let $G = T^k X$, $X \leq S_k$ primitive

$\mathrm{orbdiam}(G) = 2 \iff k = 2$ & $c(T) = 2$

so $T \cong J_1$

$\mathrm{orbdiam}(G) = 3 \iff k = 2$ & $c(T) = 3$

so T is in our list from the previous slide

$\mathrm{orbdiam}(G) = 4 \implies k = 3$ & $T \cong J_1$

or

$k = 2$ & $c(T) = 4$

in which case $\mathrm{orbdiam}(G) = 4$.

$\boxed{\text{Affine type}}$

$G = V G_0$

$\Omega = V = V_n(q)$

$G_0 \leq GL(V)$ acts irreducibly on V

$\boxed{\text{orbitals}}$ $\qquad \Gamma = \{0, a\}^G, \quad a \in V$

$\mathrm{diam}(\Gamma) = \min\{ k \mid$ every $v \in V$ is a sum of at most k vectors in $\pm a^{G_0} \}$

Lemma

$G = V G_0$ & assume G_0 contains the scalar matrices. Then $\mathrm{orbdiam}(G) \le n$.

Pf Let $\Gamma = \{0, u\}^G$ $u \in V \setminus 0$.

G_0 irred $\Rightarrow$ u^{G_0} contains a basis of V

$u_1, \dots, u_n$

$ku \in u^{G_0}$ $\forall k \in \mathbb{F}_q^*$ by assumption.

$$0 - k_1u_1 - k_1u_1 + k_2u_2 - \dots - k_1u_1 + \dots + k_nu_n$$

$\square$

examples

1. $G = V_n(q) SL_n(q)$

$SL_n(q)$ acts transitively on $V_n(q)$

$\Downarrow$

$\exists!$ non-diag orbital

$\Downarrow$

$\mathrm{orbdiam}(G) = 1$.

② $G = V_n(q)\, SO_n(q)$

orbdiam $(G) = 2$

Orbits: $\mathcal{O}_\lambda = \{ v \in V \setminus 0 : Q(v) = \lambda,\ \lambda \in \mathbb{F}_q \}$

$\Downarrow$

possibly large rank &

orbdiam (G) is low!

$\boxed{\text{Theorem (R.)}}$ G1

$G \leq AGL(V)$, G_0 almost quasisimple

V irred. in char. p

1. $G_0 \trianglerighteq A_\ell$ $\qquad \operatorname{orbdiam}(G) \geq \dfrac{\ell-5}{2\log_2 \ell}$

2. $G_0 \trianglerighteq G_\ell(r) \in \operatorname{Lie}(p')$ $\qquad \operatorname{orbdiam}(G) \geq \dfrac{r^{\ell-1}}{(\ell+1)^3 \log_2 r}$

3. $G_0 \trianglerighteq G_\ell(r) \in \operatorname{Lie}(p)$ $\qquad \operatorname{orbdiam}(G) \geq \left\lfloor \dfrac{\ell}{2} \right\rfloor$

or V is the natural module

$\boxed{\text{Theorem (R)}}$ G2

For 3. & $G_e(r)$

orbdiam $\leq 2 \iff (G_e(r), V)$ is one of the following:

$(G_2(r), V_6(r))$ r even (rank 2)

$(G_2(r), V_7(r))$ r odd (rank 4)

$(A_4(r), V_{10}(r))$ (rank 3)

$(D_5(r), V_{16}(r))$ (rank 3)

$(B_3(r), V_8(r))$ (rank 3)

$(B_4(r), V_{16}(r))$ (rank 4)

$({}^2B_2(r), V_4(r))$ r even (rank 3)

(classical, natural module)

Thank You

for

your

attention !