Elementary abelian subgroups and their local structure in classical groups

Meizheng Fu

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  Many open conjectures such as the McKay, Dade or Alperin Weight conjectures in the representation theory of finite groups can be studied by reducing to finite quasi-simple groups.

  In such studies, p-**radical subgroups** and their local structure play a critical role.

$$\underline{R} \leq G \qquad R = \underline{O_p}(\underline{N_G(R)})$$

Why need to consider elementary abelian subgroups

$G \geq H = N_G(P) \quad P \leq G.$

- Every p-radical subgroup R of G with $O_p(G) \neq G$ is radical in some maximal-proper p-local subgroup M of G.

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- Every p-radical subgroup R of G with $O_p(G) \neq G$ is radical in some maximal-proper p-local subgroup M of G.
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$$p\text{-radical subgroup } R \leqslant M = N_G(E)$$

- So it is sensible to first classify the elementary abelian subgroups of G.

- **Approach by An, Dietrich and Litterick**

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  Transfer the results to the finite group of Lie type G^F, the fixed point subgroup of G of the Steinberg endomorphism F.

$G \to G^F$

$E \to E_1, E_2, E_3$

$C_{G^F}(E_1)$

$N/C_{G^F}(E_i)$.

$c \in C_G(E)$ $\quad c\, C_G(E)^\circ$ $\quad n \in N$

Proposition 4.2 (An, Dietrich and Litterick)

$G, F, E \leqslant G$, $p \neq \ell$. $\boxed{F(n)cn^{-1}}$ $\quad c_1, c_2, c_3 \quad ncn^{-1}$

- Suppose E has a conjugate in G^F. Replacing E by this conjugate, there is a **bijection**:

$$\# \left\{ \begin{array}{c} G^F\text{-classes of} \\ \text{subgroups of } G^F \\ \text{which are } G\text{-conjugate} \\ \text{to } E \end{array} \right\} \leftrightarrow \# \left\{ \begin{array}{c} F\text{-classes in} \\ N_G(E)/C_G(E)^\circ \\ \text{contained in} \\ C_G(E)/C_G(E)^\circ \end{array} \right\} \quad (1)$$

$F(n)cn^{-1}$

$\dfrac{F(n)cn^{-1}}{ncn^{-1}}$

$E \to E_1, E_2 \quad G^F$

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What has been done

By An, Dietrich and Litterick:

What has been done

$E \leq \text{Torus } T \leq G$

By An, Dietrich and Litterick:

1. classification and local structure of toral E in *all* of the simple algebraic groups
2. classification and local structure of nontoral E in an exceptional simple algebraic group
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By Andersen et al.:

1. classification and local structure of E in the algebraic group $\mathrm{PGL}_n(\mathbb{C})$ (Theorem 8.5)

A

$n = 6 \quad 6 = 2^1 \times 3$

$r = 1 \quad k = 3$

Theorem 8.5 (Andersen et al.)

$G = \mathrm{PGL}_n(\mathbb{C})$, $n = p^r k$. $\quad p = 2 \quad \bar{T}_1 \times \bar{A} \to \mathrm{PGL}_3(\mathbb{C})$

There is a one-to-one correspondence between the conjugacy classes of the nontoral E of G, and the conjugacy classes of the toral $\bar{A}$ of $\mathrm{PGL}_k(\mathbb{C})$:

$$\left\{ \begin{array}{c} \text{nontoral } E = \\ \bar{T}_r \times \bar{A} \\ \text{in } \mathrm{PGL}_n(\mathbb{C}) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{toral} \\ \bar{A} \\ \text{in } \mathrm{PGL}_k(\mathbb{C}) \end{array} \right\}. \qquad (2)$$

$\bar{T}_r \times (\bar{A})$

T_r cld 2 of 2^{2r}

An example of $\mathrm{PGL}_6(\mathbb{C})$

Table: Nontoral elementary abelian 2-subgroups of $G = \mathrm{PGL}_6(\mathbb{C})$

Nontoral E	$C_G(E)$	$C_G(E)/C_G(E)^\circ$	$N_G(E)/C_G(E)$
$\bar{\Gamma}_1 \times$ [illegible]	$\bar{\Gamma}_1 \times \mathrm{PGL}_3(\mathbb{C})$	$\bar{\Gamma}_1$	$\mathrm{Sp}_2(2)$
$\bar{\Gamma}_1 \times 2$	$\bar{\Gamma}_1 \times (\mathrm{SL}_2(\mathbb{C}) \circ_2 T_1)$	$\bar{\Gamma}_1$	$\begin{pmatrix} \mathrm{Sp}_2(2) & 0 \\ *_{1\times 2} & 1 \end{pmatrix} \cong S_4$
$\bar{\Gamma}_1 \times 2^2$	$\bar{\Gamma}_1 \times T_2$	$\bar{\Gamma}_1$	$\begin{pmatrix} \mathrm{Sp}_2(2) & 0 \\ *_{2\times 2} & S_3 \end{pmatrix}$

What we have done

- an algorithm for the class distribution of nontoral E in $\mathrm{PGL}_n(\mathbb{C})$

What we have done

$$\mathrm{PGL}_n(\mathbb{C}) \longrightarrow \mathrm{PGL}_n(q) = \mathrm{PGL}_n(\mathbb{C})^F$$

- an algorithm for the class distribution of nontoral E in $\mathrm{PGL}_n(\mathbb{C})$
- classification and local structure of the **maximal nontoral** elementary abelian 2-subgroups in $\mathrm{PGL}_n(q)$ for q a power of a prime ℓ where $\ell \equiv 1 \pmod 4$.

What we have done

- an algorithm for the class distribution of nontoral E in $\mathrm{PGL}_n(\mathbb{C})$
- classification and local structure of the **maximal nontoral** elementary abelian 2-subgroups in $\mathrm{PGL}_n(q)$ for q a power of a prime ℓ where $\ell \equiv 1 \pmod 4$.
- classification and local structure of the **nonmaximal nontoral** elementary abelian 2-subgroups in $\mathrm{PGL}_n(q)$ for q a power of a prime ℓ where $\ell \equiv 1 \pmod 4$.

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  The **maximal nontoral** 2-subgroups

$$E = \bar{\Gamma}_r \times 2^m, m = k - 1$$

$$C_G(E) = \bar{\Gamma}_r \times T_m; \; N_G(E)/C_G(E) = \begin{pmatrix} \mathrm{Sp}_{2r}(2) & 0 \\ *_{m \times 2r} & S_{m+1} \end{pmatrix}.$$

$G \longrightarrow G^F$

$\mathrm{PGL}_n(\mathbb{C}) \longrightarrow \mathrm{PGL}_n(q)$

$q \equiv 1 (4)$

$\bar{\Gamma}_r$. $\mathrm{Sp}_{2r}(2)$

Aschba CG

What we have done

- Here are some more details.

  $G = \mathrm{PGL}_n(\mathbb{C})$, $n = 2^r k$. $r \geq 1$ and n is not a power of 2.

  The **maximal nontoral** 2-subgroups

$N = C \rtimes (N/C)$

$$E = \bar{\Gamma}_r \times 2^m, m = k - 1$$

$$C_G(E) = (\bar{\Gamma}_r \times T_m); \; N_G(E)/C_G(E) = \begin{pmatrix} \mathrm{Sp}_{2r}(2) & 0 \\ *_{m\times 2r} & S_{m+1} \end{pmatrix}.$$

$C^\circ \rtimes$

$G^F = \mathrm{PGL}_n(q)$ for q a power of a prime ℓ where $\ell \equiv 1 \pmod 4$.

We see F **centralizes**

$F(n) c n^{-1} = n c n^{-1}$

$$\boxed{N_G(E)/C_G(E)^\circ} \cong \bar{\Gamma}_r . \begin{pmatrix} \mathrm{Sp}_{2r}(2) & 0 \\ *_{m\times 2r} & S_{m+1} \end{pmatrix}$$

$\bar{\Gamma}_r \times \mathrm{Sp}_{2r}(2)$

and there are **two** F-classes in $N_G(E)/C_G(E)^\circ$ contained in $\boxed{C_G(E)/C_G(E)^\circ \cong \bar{\Gamma}_r.}$

Nonmaximal nontoral

Let $\boxed{E_{max} = \bar{\Gamma}_r \times \bar{A}_{max}}$ be a maximal nontoral elementary abelian 2-subgroup of G where $\bar{A}_{max}$ is maximal toral in $\mathrm{PGL}_k(\mathbb{C})$ and has rank m where $m = k - 1$. Let $\boxed{E_1 = \bar{\Gamma}_r \times \bar{A}_{in}}$ be a nontoral elementary abelian 2-subgroup of G where $\bar{A}_{in} < \bar{A}_{max}$. Here

$\bar{A}_{in}$ $GL_k(\mathbb{C})$

$$N_G(E_1)/C_G(E_1)^{\circ} = \bar{\Gamma}_r \times C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A}_{in})/C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A}_{in})^{\circ}.\begin{pmatrix} \mathrm{Sp}_{2r}(2) & 0 \\ *_{rk\bar{A}_{in}\times 2r} & W_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A}_{in}) \end{pmatrix},$$

$$N_G(E_{max})/C_G(E_{max})^{\circ} = \bar{\Gamma}_r.\begin{pmatrix} \mathrm{Sp}_{2r}(2) & 0 \\ *_{m\times 2r} & S_{m+1} \end{pmatrix}.$$

$\bar{\Gamma}_r$

, acts trivially

What we have done

Theorem

In $G = \mathrm{PGL}_n(\mathbb{C})$ where $n = 2^r k$ and $r \geq 1$, let $E_1 = \bar{\Gamma}_r \times \bar{A}_{in}$ be a nontoral elementary abelian 2-subgroup and let $E_{max} = \bar{\Gamma}_r \times \bar{A}_{max}$ be a maximal nontoral elementary abelian 2-subgroup. Here $\bar{A}_{max}$, maximal toral in $\mathrm{PGL}_k(\mathbb{C})$, has rank m where $m = k - 1$ and $\bar{A}_{in} < \bar{A}_{max}$.

There exists $U_1 \leq N_1$ such that $u \in N_G(E_1) \setminus C_G(E_1)$ for each nontrivial $u \in U_1$ and $U_1 C_G(E_1)/C_G(E_1)$ is the subgroup U_{in} of $W_G(E_1)$.

What we have done

Consequently, as is in the maximal nontoral case, $N_G(E_1)/C_G(E_1)^\circ$ is **centralised** by F for every nonmaximal nontoral elementary abelian 2-subgroup E_1 of G and for q a power of a prime ℓ where $\ell \equiv 1 \pmod 4$.

What we have done

We next descend to the finite groups. $G \longrightarrow G^F$

Case 1. $\bar{A}$ **is nontrivial nonmaximal toral in** $\mathrm{PGL}_k(\mathbb{C})$ **with** $C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A})$ **connected.**

$$E = \bar{\Gamma}_r \times \bar{A}$$

$$C_G(E) = \bar{\Gamma}_r \times C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A})$$

$$N_G(E)/C_G(E) \cong \begin{pmatrix} \mathrm{Sp}_{2r}(2) & 0 \\ *_{rk\bar{A}_{in} \times 2r} & W_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A}_{in}) \end{pmatrix}$$

$$C_G(E)/C_G(E)^\circ \cong \bar{\Gamma}_r$$

Γ_r. $Sp(2)$

$$N_G(E)/C_G(E)^\circ \cong \bar{\Gamma}_r \begin{pmatrix} \mathrm{Sp}_{2r}(2) & 0 \\ *_{rk\bar{A}_{in} \times 2r} & W_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A}_{in}) \end{pmatrix}$$

What we have done

Case 1. $\bar{A}$ is nontrivial nonmaximal toral in $\mathrm{PGL}_k(\mathbb{C})$ with $C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A})$ connected.

Hence there are two F-classes of $N_G(E)/C_G(E)^\circ$ in $C_G(E)/C_G(E)^\circ$, and correspondingly, there are two G^F-conjugacy classes of elementary abelian 2-subgroups.

Case 2. $\bar{A}$ **is nontrivial nonmaximal toral in** $\mathrm{PGL}_k(\mathbb{C})$ **with** $C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A})$ **disconnected.**

$\to 2^{2r}$

$\bar{\Gamma}_r . Sp_{2r}(2)$

$$C_G(E)/C_G(E)^\circ \cong \bar{\Gamma}_r \times C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A})/C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A})^\circ$$

$$N_G(E)/C_G(E)^\circ \cong \bar{\Gamma}_r \times C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A})/C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A})^\circ . \begin{pmatrix} \mathrm{Sp}_{2r}(2) & 0 \\ *_{rk\bar{A}\times 2r} & W_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A}) \end{pmatrix}$$

$C/C^\circ = \bar{\Gamma}_r \times B_r$

$B_r = 2^r$

$\leq *_{m\times 2r}$

$C/C^\circ = 2^r$

What we have done

Theorem

For $n = 2^s \times t$ with $\gcd(2, t) = 1$ and $s \geq 1$, the conjugacy classes of the toral elementary abelian 2-subgroups of $G = \mathrm{PGL}_n(\mathbb{C})$ with disconnected centralisers have representatives $D = D_r \times \bar{A}$ for $1 \leq r \leq s$, $n = 2^r \times k$ and $\bar{A}$ is trivial or a representative of a conjugacy class of the toral elementary abelian 2-subgroups of $\mathrm{PGL}_k(\mathbb{C})$ with $C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A})$ connected.

And

$$C_G(D)/C_G(D)^\circ \cong B_r.$$

$$N_G(D)/C_G(D) \cong \left(\begin{pmatrix} \mathrm{GL}_r(2) & 0 \\ *_{\mathrm{rk}\bar{A} \times r} & W_{\mathrm{PGL}_{2^{s-r}t}(\mathbf{C})}(\bar{U}_1) \end{pmatrix}\right) \to 2 \text{ orbits}$$

Br.

2^r

What we have done

There are three F-classes of $N_G(E)/C_G(E)^\circ$ in $C_G(E)/C_G(E)^\circ$ and correspondingly, there are three G^F-conjugacy classes of elementary abelian 2-subgroups.

$\underline{C}$, $\underline{N|C}$

$PGL_n(\mathbb{C})$ $\underline{3}$

odd

$C = C^\circ$

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$\bar{\Gamma}_1 \times 2$	$\bar{\Gamma}_1 \times (SL_2(\mathbb{C}) \circ_2 T_1)$	$\bar{\Gamma}_1$	$\begin{pmatrix} Sp_2(2) & 0 \\ *_{1\times 2} & 1 \end{pmatrix} \cong S_4$
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$E_1 =$, E_2, E_m

2, 2, 2

$2^2 \leq PGL_3$

2

max

What we will do next

$\bar{F}(n) c n^{-1}$

$\overline{T}r_{-} Sp_{2r}(2) \leq \bar{F}$

- Extend the condition of q being a power of a prime ℓ where $\cancel{\ell \equiv 1}$ ~~(mod 4) to~~ $\ell \equiv 3 \pmod 4$.

What we will do next

dal 2-gps

- Extend the condition of q being a power of a prime ℓ where $\ell \equiv 1 \pmod 4$ to $\ell \equiv 3 \pmod 4$.
- Extend to the elementary abelian p-subgroups of classical groups of type A for p odd.

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- Extend the condition of q being a power of a prime ℓ where $\ell \equiv 1 \pmod 4$ to $\ell \equiv 3 \pmod 4$.
- Extend to the elementary abelian p-subgroups of classical groups of type A for p odd.
- Explore whether the above method can be applied to classify the elementary abelian 2-subgroups and the local structure in classical groups of type C; if not, then establish new methods to accomplish this.

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- Extend the condition of q being a power of a prime ℓ where $\ell \equiv 1 \pmod 4$ to $\ell \equiv 3 \pmod 4$.
- Extend to the elementary abelian p-subgroups of classical groups of type A for p odd.
- Explore whether the above method can be applied to classify the elementary abelian 2-subgroups and the local structure in classical groups of type C; if not, then establish new methods to accomplish this.
- Classify the elementary abelian p-subgroups and the local structure in classical groups of types B and D.

Thank you for listening!