

Elementary abelian subgroups and their local structure in classical groups

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Why need to consider elementary abelian subgroups

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Many open conjectures such as the McKay, Dade or Alperin Weight conjectures in the representation theory of finite groups can be studied by reducing to finite quasi-simple groups.

In such studies, p -radical subgroups and their local structure play a critical role.

$$\underline{R} \leq G \quad R = \underline{O}_p(N_G(R))$$

Why need to consider elementary abelian subgroups

$$G \ni H = N_G(P) \quad P \leq G.$$

- Every p -radical subgroup R of G with $O_p(G) \neq G$ is radical in some maximal-proper p -local subgroup M of G .

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$$p\text{-radical subgroup } R \leq M = N_G(E)$$

- So it is sensible to first classify the elementary abelian subgroups of G .

How to classify E in finite groups

- **Approach by An, Dietrich and Litterick**

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Classify in a linear algebraic group G

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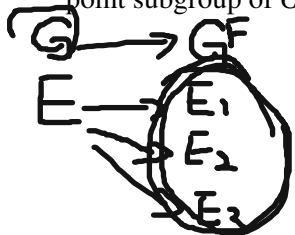
How to classify E in finite groups

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Classify in a linear algebraic group G .



Transfer the results to the finite group of Lie type G^F , the fixed point subgroup of G of the Steinberg endomorphism F .



$$C_{G^F}(E_i)$$

$$N/C_{G^F}(E_i).$$

$C_G(E) \subset C_G(E)^\circ$ $\rightarrow n \in N$

Proposition 4.2 (An, Dietrich and Litterick)

$G, F, E \leq G, p \neq \ell$. $F(n)cn^{-1}$

- Suppose E has a conjugate in G^F . Replacing E by this conjugate, there is a bijection:

$$\frac{F(n)cn^{-1}}{ncn^{-1}} \left\{ \begin{array}{l} G^F\text{-classes of} \\ \text{subgroups of } G^F \\ \text{which are } G\text{-conjugate} \\ \text{to } E \end{array} \right\} \# = \# \frac{F(n)cn^{-1}}{ncn^{-1}} \left\{ \begin{array}{l} F\text{-classes in} \\ N_G(E)/C_G(E)^\circ \\ \text{contained in} \\ C_G(E)/C_G(E)^\circ \end{array} \right\} \# \dots (1)$$



~~*~~

What has been done

By An, Dietrich and Litterick:

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$$E \leq \text{Torus} \leq \underline{\underline{I}} \leq G$$

By An, Dietrich and Litterick:

1. classification and local structure of toral E in all of the simple algebraic groups

2. classification and local structure of nontoral E in an exceptional simple algebraic group

3. classification and local structure of E in finite exceptional groups of Lie type *

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By Andersen et al.:

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1. classification and local structure of toral E in *all* of the simple algebraic groups

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By Andersen et al.:

1. classification and local structure of E in the algebraic group $\mathrm{PGL}_n(\mathbb{C})$ (Theorem 8.5)

A

$$n = 6$$

$$l = 2 \times 3$$

$$r = 1 \quad k = 3$$

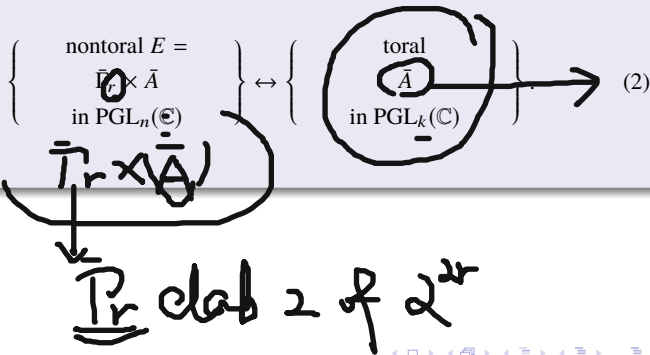
Theorem 8.5 (Andersen et al.)

$G = \text{PGL}_n(\mathbb{C}), n = p^r k.$

$$p = 2$$

$$\bar{\Gamma}_1 \times \bar{A} \rightarrow \text{PGL}_3(\mathbb{C})$$

There is a one-to-one correspondence between the conjugacy classes of the nontoral E of G , and the conjugacy classes of the toral \bar{A} of $\text{PGL}_k(\mathbb{C})$:



An example of $\mathrm{PGL}_6(\mathbb{C})$

Table: Nontoral elementary abelian 2-subgroups of $G = \mathrm{PGL}_6(\mathbb{C})$

Nontoral H	$C_G(E)$	$C_G(E)/C_G(E)^\circ$	$N_G(E)/C_G(E)$
$\bar{\Gamma}_1 \times T_1$	$\bar{\Gamma}_1 \times \mathrm{PGL}_3(\mathbb{C})$	$\bar{\Gamma}_1$	$\mathrm{Sp}_2(2)$
$\bar{\Gamma}_1 \times 2$	$\times (\mathrm{SL}_2(\mathbb{C}) \circ_2 T_1)$	$\bar{\Gamma}_1$	$\begin{pmatrix} \mathrm{Sp}_2(2) & 0 \\ *_{1 \times 2} & 1 \end{pmatrix} \cong S_4$
$\bar{\Gamma}_1 \times 2^2$	$\bar{\Gamma}_1 \times T_2$	$\bar{\Gamma}_1$	$\begin{pmatrix} \mathrm{Sp}_2(2) & 0 \\ *_{2 \times 2} & S_3 \end{pmatrix}$

What we have done

- an algorithm for the class distribution of nontoral E in $\mathrm{PGL}_n(\mathbb{C})$

What we have done

$$\mathrm{PGL}_n(\mathbb{C}) \longrightarrow \mathrm{PGL}_n(q) \stackrel{=}{=} \mathrm{PGL}_n(\mathbb{C})^F$$

- an algorithm for the class distribution of nontoral E in $\mathrm{PGL}_n(\mathbb{C})$
- classification and local structure of the **maximal nontoral** elementary abelian 2-subgroups in $\mathrm{PGL}_n(q)$ for q a power of a prime ℓ where $\ell \equiv 1 \pmod{4}$.

What we have done

- an algorithm for the class distribution of nontoral E in $\mathrm{PGL}_n(\mathbb{C})$
- classification and local structure of the **maximal nontoral** elementary abelian 2-subgroups in $\mathrm{PGL}_n(q)$ for q a power of a prime ℓ where $\ell \equiv 1 \pmod{4}$.
- classification and local structure of the **nonmaximal nontoral** elementary abelian 2-subgroups in $\mathrm{PGL}_n(q)$ for q a power of a prime ℓ where $\ell \equiv 1 \pmod{4}$.

What we have done

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$G = \text{PGL}_n(\mathbb{C})$, $n = 2^r k$. $r \geq 1$ and n is not a power of 2.

$k \neq 1$

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$G = \mathrm{PGL}_n(\mathbb{C})$, $n = 2^r k$. $r \geq 1$ and n is not a power of 2.

The **maximal nontoral** 2-subgroups

$$E = \bar{\Gamma}_r \times 2^m, m = k - 1$$

$$\underline{C_G(E)} = \bar{\Gamma}_r \times T_m; \quad \underline{N_G(E)/C_G(E)} = \begin{pmatrix} \mathrm{Sp}_{2r}(2) & 0 \\ *_{m \times 2r} & S_{m+1} \end{pmatrix}.$$

$$G \longrightarrow G^F$$

$$\mathrm{PGL}_n(\mathbb{C}) \longrightarrow \mathrm{PGL}_n(\mathbb{F}_q)$$

$\bar{\Gamma}_r, \mathrm{Sp}_{2r}(2)$
 $\text{Atschba } \mathbb{C} \mathbb{G}$

$q=1(4)$

What we have done

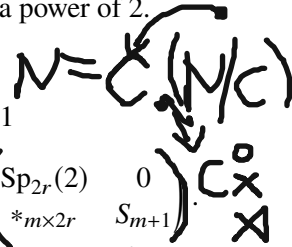
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The **maximal nontoral** 2-subgroups

$$E = \bar{\Gamma}_r \times 2^m, m = k - 1$$

$$C_G(E) = \left(\bar{\Gamma}_r \times T_m \right) \cdot N_G(E)/C_G(E) = \left(\begin{array}{cc} \mathrm{Sp}_{2r}(2) & 0 \\ *_{m \times 2r} & S_{m+1} \end{array} \right) \cdot C \times \dots$$

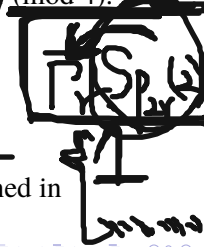
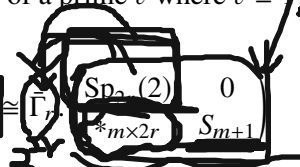


$G^F = \mathrm{PGL}_n(q)$ for q a power of a prime ℓ where $\ell \equiv 1 \pmod{4}$.

We see F centralizes

$$F(n) \subset \Gamma^{-1} \\ = n \subset n^{-1}$$

$$N_G(E)/C_G(E)^\circ \cong \bar{\Gamma}_r$$



and there are two F classes in $N_G(E)/C_G(E)^\circ$ contained in

$$C_G(E)/C_G(E)^\circ \cong \bar{\Gamma}_r.$$

What we have done

Nonmaximal nontoral

Let $E_{max} = \bar{\Gamma}_r \times \bar{A}_{max}$ be a maximal nontoral elementary abelian 2-subgroup of G where \bar{A}_{max} is maximal toral in $\text{PGL}_k(\mathbb{C})$ and has rank m where $m = k - 1$. Let $E_1 = \bar{\Gamma}_r \times \bar{A}_{in}$ be a nontoral elementary abelian 2-subgroup of G where $\bar{A}_{in} < \bar{A}_{max}$. Here

$$F \curvearrowright (\mathbb{C}/\mathbb{C}^\circ) \cdot (N_k)$$

$$\bar{A}_{in} \curvearrowright \text{PGL}_k(\mathbb{C})$$

F

$$\bar{\Gamma}_r \times \frac{N_G(E_1)/C_G(E_1)^\circ}{\text{CPGL}_k(\mathbb{C})(\bar{A}_{in})/\text{CPGL}_k(\mathbb{C})(\bar{A}_{in})^\circ} \cong \begin{pmatrix} \text{Sp}_{2r}(2) & 0 \\ *_{rk\bar{A}_{in} \times 2r} & W_{\text{PGL}_k(\mathbb{C})(\bar{A}_{in})} \end{pmatrix}$$

$$N_G(E_{max})/C_G(E_{max})^\circ = \bar{\Gamma}_r \cdot \begin{pmatrix} \text{Sp}_{2r}(2) & 0 \\ *_{m \times 2r} & S_{m+1} \end{pmatrix}$$

acts faithfully

Theorem

In $G = \mathrm{PGL}_n(\mathbb{C})$ where $n = 2^r k$ and $r \geq 1$, let $E_1 = \bar{\Gamma}_r \times \bar{A}_{in}$ be a nontoral elementary abelian 2-subgroup and let $E_{max} = \bar{\Gamma}_r \times \bar{A}_{max}$ be a maximal nontoral elementary abelian 2-subgroup. Here \bar{A}_{max} , maximal toral in $\mathrm{PGL}_k(\mathbb{C})$, has rank m where $m = k - 1$ and $\bar{A}_{in} < \bar{A}_{max}$.

There exists $U_1 \leq N_1$ such that $u \in N_G(E_1) \setminus C_G(E_1)$ for each nontrivial $u \in U_1$ and $U_1 C_G(E_1) / C_G(E_1)$ is the subgroup U_{in} of $W_G(E_1)$.

What we have done

Consequently, as is in the maximal nontoral case, $N_G(E_1)/C_G(E_1)^\circ$ is **centralised** by F for every nonmaximal nontoral elementary abelian 2-subgroup E_1 of G and for q a power of a prime ℓ where $\ell \equiv 1 \pmod{4}$.

What we have done

We next descend to the finite groups.

$$G \longrightarrow G^F$$

Case 1. \bar{A} is nontrivial nonmaximal toral in $\mathrm{PGL}_k(\mathbb{C})$ with $C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A})$ connected.

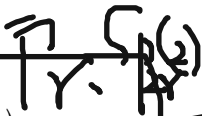
$$E = \bar{\Gamma}_r \times \bar{A}$$

$$C_G(E) = \bar{\Gamma}_r \times C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A})$$

$$N_G(E)/C_G(E) \cong \begin{pmatrix} \mathrm{Sp}_{2r}(2) & 0 \\ *_{rk\bar{A}_{in} \times 2r} & W_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A}_{in}) \end{pmatrix}$$

$$C_G(E)/C_G(E)^\circ \cong \bar{\Gamma}_r$$

$$N_G(E)/C_G(E)^\circ \cong \begin{pmatrix} \bar{\Gamma}_r & \mathrm{Sp}_{2r}(2) & 0 \\ *_{rk\bar{A}_{in} \times 2r} & W_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A}_{in}) & \end{pmatrix}$$



Case 1. \bar{A} is nontrivial nonmaximal toral in $\mathrm{PGL}_k(\mathbb{C})$ with $C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A})$ connected.

Hence there are two F -classes of $N_G(E)/C_G(E)^\circ$ in $C_G(E)/C_G(E)^\circ$, and correspondingly, there are two G^F -conjugacy classes of elementary abelian 2-subgroups.

What we have done

Case 2. \bar{A} is nontrivial nonmaximal toral in $\mathrm{PGL}_k(\mathbb{C})$ with

$C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A})$ disconnected.

$\rightarrow 2^{2r}$

$\bar{\Gamma}_r \cdot \mathrm{Sp}_{2r}(2)$

$$C_G(E)/C_G(E)^\circ \cong \bar{\Gamma}_r \times C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A})/C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A})^\circ$$

$$N_G(E)/C_G(E)^\circ \cong \bar{\Gamma}_r \times \left[\cancel{C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A})/C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A})^\circ} \right] \times \left[\begin{array}{c} \mathrm{Sp}_{2r}(2) \\ *rk\bar{A} \times 2r \end{array} \right] \times \left[\begin{array}{c} 0 \\ \cancel{W_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A})} \end{array} \right]$$

$$\begin{aligned} \mathbb{C}/\mathbb{C}^\circ \\ = \bar{\Gamma}_r \times B_r \end{aligned}$$

$B_r = 2^r$

$\cong *_{m \times 2r}$

$\mathbb{C}/\mathbb{C}^\circ = 2^r$

Theorem

For $n = 2^s \times t$ with $\gcd(2, t) = 1$ and $s \geq 1$, the conjugacy classes of the toral elementary abelian 2-subgroups of $G = \mathrm{PGL}_n(\mathbb{C})$ with disconnected centralisers have representatives $D = D_r \times \bar{A}$ for $1 \leq r \leq s$, $n = 2^r \times k$ and \bar{A} is trivial or a representative of a conjugacy class of the toral elementary abelian 2-subgroups of $\mathrm{PGL}_k(\mathbb{C})$ with $C_{\mathrm{PGL}_k(\mathbb{C})}(\bar{A})$ connected.

And

$$C_G(D)/C_G(D)^\circ \cong B_r.$$

$$N_G(D)/C_G(D) \sim \left(\begin{array}{c|c} \mathrm{GL}_r(2) & 0 \\ \hline \mathrm{PGL}_{k \times r}(\bar{A}) & W_{\mathrm{PGL}_{2^{s-r}t}(\mathbb{C})}(\bar{U}_1) \end{array} \right) \rightarrow 2 \text{ orbits}$$

Br.
↓
 2^r

What we have done

There are three F -classes of $N_G(E)/C_G(E)^\circ$ in $C_G(E)/C_G(E)^\circ$ and correspondingly, there are three G^F -conjugacy classes of elementary abelian 2-subgroups.

An example of $\text{PGL}_6(\mathbb{C})$

\mathbb{C} , N/C

$\text{PGL}_n(\mathbb{C})$

\downarrow odd

$C = \mathbb{C}^\circ$

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$\bar{\Gamma}_1 \times 2$	$\bar{\Gamma}_1 \times (\text{SL}_2(\mathbb{C}) \circ_2 T_1)$	$\bar{\Gamma}_1$	$\begin{pmatrix} \text{Sp}_2(2) & 0 \\ *_{1 \times 2} & 1 \end{pmatrix} \cong S_4$
$\bar{\Gamma}_1 \times 2^2$	$\bar{\Gamma}_1 \times T_2$	$\bar{\Gamma}_1$	$\begin{pmatrix} \text{Sp}_2(2) & 0 \\ *_{2 \times 2} & S_3 \end{pmatrix}$

$\leftarrow \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

$\mathbb{Z}^2 \leq \text{PGL}_3$

\mathbb{Z}

$\rightarrow \text{max}$

What we will do next

$$\overline{F}(n)cn^{-1}$$

$$\overline{\text{Tr-Sp}}_{2r}(2) \leftarrow F$$

- Extend the condition of q being a power of a prime ℓ where $\ell \equiv 1 \pmod{4}$ to $\ell \equiv 3 \pmod{4}$.

What we will do next

do 2-eps

- Extend the condition of q being a power of a prime ℓ where $\ell \equiv 1 \pmod{4}$ to $\ell \equiv 3 \pmod{4}$.
- Extend to the elementary abelian p -subgroups of classical groups of type A for p odd.

What we will do next

- Extend the condition of q being a power of a prime ℓ where $\ell \equiv 1 \pmod{4}$ to $\ell \equiv 3 \pmod{4}$.
- Extend to the elementary abelian p -subgroups of classical groups of type A for p odd.
- Explore whether the above method can be applied to classify the elementary abelian 2-subgroups and the local structure in classical groups of type C ; if not, then establish new methods to accomplish this.

What we will do next

- Extend the condition of q being a power of a prime ℓ where $\ell \equiv 1 \pmod{4}$ to $\ell \equiv 3 \pmod{4}$.
- Extend to the elementary abelian p -subgroups of classical groups of type A for p odd.
- Explore whether the above method can be applied to classify the elementary abelian 2-subgroups and the local structure in classical groups of type C ; if not, then establish new methods to accomplish this.
- Classify the elementary abelian p -subgroups and the local structure in classical groups of types B and D .

Thank you for listening!