Regular Maps

Darius Young

February 18, 2023

Contents

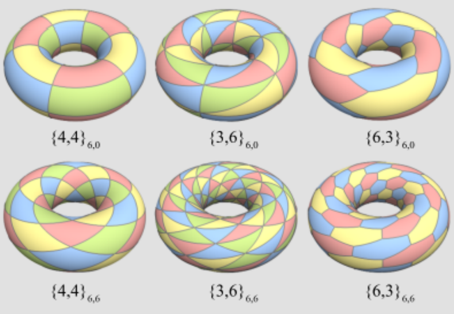
Introduction

Introduction

A regular map at first glance is a graph embedded in a compact $2-$manifold, such that every face is homeomorphic to a disk and its automorphism group acts regularly on flags (edge, vertex, face tuples). Here regularly means that there is exactly one automorphism mapping any flag to any other.

Introduction

A regular map at first glance is a graph embedded in a compact $2-$manifold, such that every face is homeomorphic to a disk and its automorphism group acts regularly on flags (edge, vertex, face tuples). Here regularly means that there is exactly one automorphism mapping any flag to any other.

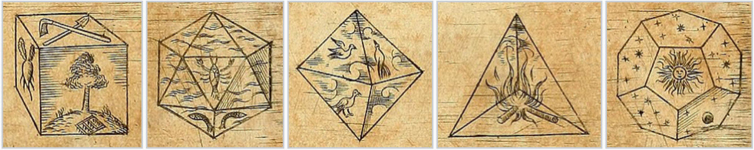


Examples

Likely the first examples found were the platonic solids which are all regular maps on the sphere.

Examples

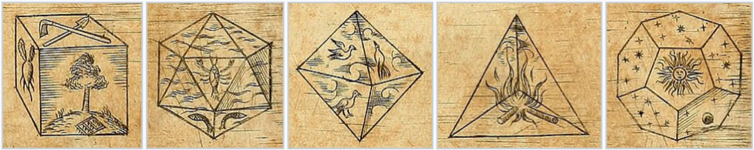
Likely the first examples found were the platonic solids which are all regular maps on the sphere.



Regularity implies we can associate a unique automorphism to every flag of the map, so maps are in a way the Cayley graph of their own automorphism group. Furthermore any maps automorphism group is 3 generated by reflections.

Examples

Likely the first examples found were the platonic solids which are all regular maps on the sphere.



Regularity implies we can associate a unique automorphism to every flag of the map, so maps are in a way the Cayley graph of their own automorphism group. Furthermore any maps automorphism group is 3 generated by reflections.

So given a map we can construct its automorphism group and given the group we can construct the associated map.

Triangle groups

Regular maps have another description in terms of triangle groups. Which are groups generated by 3 reflections in the plane of angles $(\frac{\pi}{l}, \frac{\pi}{k}, \frac{\pi}{m})$ so we denote such a group by

Triangle groups

Regular maps have another description in terms of triangle groups. Which are groups generated by 3 reflections in the plane of angles $(\frac{\pi}{l}, \frac{\pi}{k}, \frac{\pi}{m})$ so we denote such a group by

$$\triangle(l,k,m) = \langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^l = (yz)^k = (zx)^m = 1 \rangle$$

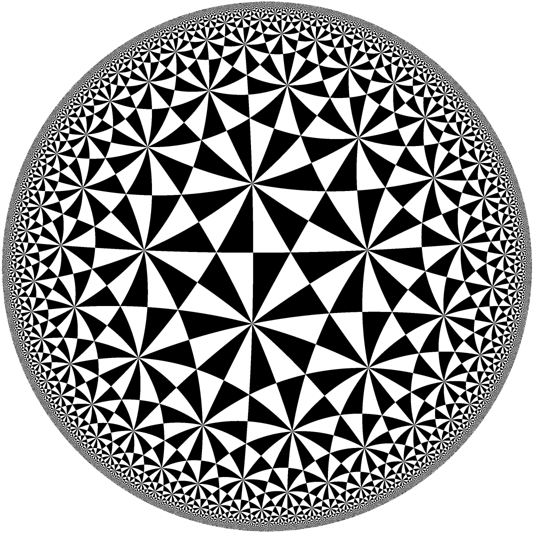
Triangle groups

Regular maps have another description in terms of triangle groups. Which are groups generated by 3 reflections in the plane of angles $(\frac{\pi}{l}, \frac{\pi}{k}, \frac{\pi}{m})$ so we denote such a group by

$$\triangle(l,k,m) = \langle x,y,z \mid x^2 = y^2 = z^2 = (xy)^l = (yz)^k = (zx)^m = 1\rangle$$

These groups fall into 3 categories. Spherical, Euclidean or Hyperbolic. This corresponds to whether $\frac{1}{k} + \frac{1}{l} + \frac{1}{m}$ is greater than (Spherical) equal to (Elliptic) or less than 1 (Hyperbolic).

Triangle groups



Triangle group quotients are maps

Now take a triangle group $\triangle(2, m, k)$ i.e a right triangle tiling of some space. A regular map is simply a quotient of such a group by a torsion free normal subgroup which preserves the orders m and k.

Triangle group quotients are maps

Now take a triangle group $\triangle(2, m, k)$ i.e a right triangle tiling of some space. A regular map is simply a quotient of such a group by a torsion free normal subgroup which preserves the orders m and k.

To explicitly construct the edge and vertex connections we take the right cosets of the images of $\langle xy \rangle$, $\langle yz \rangle$, $\langle zx \rangle$ in the quotient with incidence given by non empty intersection.

Triangle group quotients are maps

Now take a triangle group $\triangle(2, m, k)$ i.e a right triangle tiling of some space. A regular map is simply a quotient of such a group by a torsion free normal subgroup which preserves the orders m and k.

To explicitly construct the edge and vertex connections we take the right cosets of the images of $\langle xy \rangle$, $\langle yz \rangle$, $\langle zx \rangle$ in the quotient with incidence given by non empty intersection.

From this perspective the theory of regular maps is exactly equivalent to the theory of "smooth" quotients of $\{2, m, k\}$ triangle groups. Here smooth means that the quotient preserves the order of yz, zx, xy.

$$G = \langle x, y, z | x^2 = y^2 = z^2 = (yz)^k = (zx)^m = (xy)^2 = \cdots = 1 \rangle$$

Reflexible and Chiral

Let M be a map then we denote its automorphism group by $Aut(M)$ and we denote its orientation preserving automorphisms by $Aut^+(M)$. Note that $Aut^+(M)$ has index at most 2 in $Aut(M)$.

Reflexible and Chiral

Let M be a map then we denote its automorphism group by $Aut(M)$ and we denote its orientation preserving automorphisms by $Aut^+(M)$. Note that $Aut^+(M)$ has index at most 2 in $Aut(M)$.

Sometimes maps admit an automorphism which is a reflection, i.e non orientation preserving then $[Aut(M) : Aut^+(M)] = 2$ and we call such a map reflexible otherwise we call it chiral.

A useful formula

We now derive a necessary condition for a $\{k, m\}$ map to exist

A useful formula

We now derive a necessary condition for a $\{k, m\}$ map to exist

$$|G| = 4E = 2mF = 2kV$$

A useful formula

We now derive a necessary condition for a $\{k, m\}$ map to exist

$$|G| = 4E = 2mF = 2kV$$

Euler's formula $2 - 2g = V - E + F$ gives us the equation,

$$2 - 2g = \frac{|G|}{2}\left(\frac{1}{k} + \frac{1}{m} - \frac{1}{2}\right)$$

A useful formula

We now derive a necessary condition for a $\{k, m\}$ map to exist

$$|G| = 4E = 2mF = 2kV$$

Euler's formula $2 - 2g = V - E + F$ gives us the equation,

$$2 - 2g = \frac{|G|}{2}\left(\frac{1}{k} + \frac{1}{m} - \frac{1}{2}\right)$$

Or for the orientation preserving case

$$2 - 2g = |G^+|\left(\frac{1}{k} + \frac{1}{m} - \frac{1}{2}\right)$$

A special family of maps

Hurwitz maps

To get some examples we can look at one interesting family of maps, the so called Hurwitz maps. These come from the realisation by Hurwitz that the order of the conformal orientation preserving automorphism group of any surface of genus g is bounded above by $84(g-1)$.

Hurwitz maps

To get some examples we can look at one interesting family of maps, the so called Hurwitz maps. These come from the realisation by Hurwitz that the order of the conformal orientation preserving automorphism group of any surface of genus g is bounded above by $84(g-1)$.

$$2 - 2g = |G^{+}| \left(\frac{1}{k} + \frac{1}{m} - \frac{1}{2} \right)$$

Hurwitz maps

To get some examples we can look at one interesting family of maps, the so called Hurwitz maps. These come from the realisation by Hurwitz that the order of the conformal orientation preserving automorphism group of any surface of genus g is bounded above by $84(g-1)$.

$$2-2g = |G^{+}|\left(\frac{1}{k}+\frac{1}{m}-\frac{1}{2}\right)$$

$$(2-2g)\left(\frac{1}{k}+\frac{1}{m}-\frac{1}{2}\right)^{-1} = |G^{+}|$$

Hurwitz maps

To get some examples we can look at one interesting family of maps, the so called Hurwitz maps. These come from the realisation by Hurwitz that the order of the conformal orientation preserving automorphism group of any surface of genus g is bounded above by $84(g-1)$.

$$2 - 2g = |G^+| \left(\frac{1}{k} + \frac{1}{m} - \frac{1}{2} \right)$$

$$(2-2g) \left(\frac{1}{k} + \frac{1}{m} - \frac{1}{2} \right)^{-1} = |G^+|$$

Now $\frac{1}{2} - \frac{1}{k} - \frac{1}{m}$ is minimised when $m = 3$ and $k = 7$ when its value is $\frac{1}{42}$ hence,

Hurwitz maps

To get some examples we can look at one interesting family of maps, the so called Hurwitz maps. These come from the realisation by Hurwitz that the order of the conformal orientation preserving automorphism group of any surface of genus g is bounded above by $84(g-1)$.

$$2 - 2g = |G^+| \left(\frac{1}{k} + \frac{1}{m} - \frac{1}{2} \right)$$

$$(2 - 2g) \left(\frac{1}{k} + \frac{1}{m} - \frac{1}{2} \right)^{-1} = |G^+|$$

Now $\frac{1}{2} - \frac{1}{k} - \frac{1}{m}$ is minimised when $m = 3$ and $k = 7$ when its value is $\frac{1}{42}$ hence,

$$84(g-1) \leq |G^+|$$

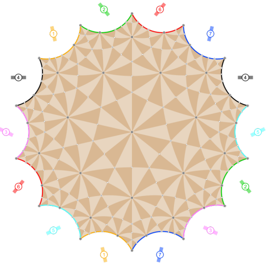
Finding Hurwitz maps

Hurwitz maps are therefore the maps for which this bound is attained, i.e when $|G^+| = 84(g-1)$. But do any even exist?

Finding Hurwitz maps

Hurwitz maps are therefore the maps for which this bound is attained, i.e when $|G^+| = 84(g-1)$. But do any even exist?

Yes! the smallest genus Hurwitz map is the automorphism group of the Klein quartic, given by the zero set of $x^3y + y^3z + z^3x$. This group has order $168 = 84(3-1)$ so is genus 3.



$$\langle R, S \mid R^3 = (RS)^2 = S^7 = (RS^{-2})^4 = 1 \rangle$$

Finding more Huwitz maps

Finding more Huwitz maps

Let G be a Hurwitz group of genus g which corresponds to a normal subgroup K of index $84(g-1)$ in $\triangle(2,3,7)$. Then take the subgroup L$=[K,K]\cdot K^m$ this is the subgroup generated by commutators and m^{th} powers in K. This subgroup is clearly characteristic (invariant under automorphism) and therefore is normal in $\triangle(2,3,7)$.

Finding more Huwitz maps

Let G be a Hurwitz group of genus g which corresponds to a normal subgroup K of index $84(g-1)$ in $\triangle(2,3,7)$. Then take the subgroup L$=[K,K]\cdot K^m$ this is the subgroup generated by commutators and m^{th} powers in K. This subgroup is clearly characteristic (invariant under automorphism) and therefore is normal in $\triangle(2,3,7)$.

Since K must be a surface group it has presentation in terms of $2g$ generators A_i, B_i for $i \in \{1,\ldots,g\}$ with one relation $\prod_{i\leq g}[A_i, B_i]$. Quotenting by L has the effect of making these generators commute and have order m so

Finding more Huwitz maps

Let G be a Hurwitz group of genus g which corresponds to a normal subgroup K of index $84(g-1)$ in $\triangle(2,3,7)$. Then take the subgroup L=$[K,K]\cdot K^m$ this is the subgroup generated by commutators and m^{th} powers in K. This subgroup is clearly characteristic (invariant under automorphism) and therefore is normal in $\triangle(2,3,7)$.

Since K must be a surface group it has presentation in terms of $2g$ generators A_i, B_i for $i \in \{1,\ldots,g\}$ with one relation $\prod_{i\leq g}[A_i,B_i]$. Quotenting by L has the effect of making these generators commute and have order m so

$$K/L \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \times \cdots \times \mathbb{Z}/m\mathbb{Z}$$

where there are $2g$ factors of $\mathbb{Z}/m\mathbb{Z}$. So $[K:L]=m^{2g}$.

Finding more Huwitz maps

Let G be a Hurwitz group of genus g which corresponds to a normal subgroup K of index $84(g-1)$ in $\triangle(2,3,7)$. Then take the subgroup L=$[K,K]\cdot K^m$ this is the subgroup generated by commutators and m^{th} powers in K. This subgroup is clearly characteristic (invariant under automorphism) and therefore is normal in $\triangle(2,3,7)$.

Since K must be a surface group it has presentation in terms of $2g$ generators A_i, B_i for $i \in \{1,\ldots,g\}$ with one relation $\prod_{i\leq g}[A_i,B_i]$. Quotenting by L has the effect of making these generators commute and have order m so

$$K/L \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \times \cdots \times \mathbb{Z}/m\mathbb{Z}$$

where there are $2g$ factors of $\mathbb{Z}/m\mathbb{Z}$. So $[K:L]=m^{2g}$. Meaning that also $[\triangle(2,3,7):L]=84(g-1)\cdot m^{2g}$ hence

$$|\triangle(2,3,7)/L| = 84(m^{2g}(g-1)+1)$$

The same trick for soluble maps

The above technique called the 'Macbeath Trick' & is surprisingly useful, it can also be used to show that there are for example infinitely many soluble maps.

The same trick for soluble maps

The above technique called the 'Macbeath Trick' & is surprisingly useful, it can also be used to show that there are for example infinitely many soluble maps.

Let G be a soluble regular map of genus g which corresponds to a normal subgroup K of index n in $T = \triangle(2, m, k)$. Then again take the subgroup $L = [K, K] \cdot K^d$. Again we have that K/L is abelian (and therefore soluble) and T/L is a regular map.

The same trick for soluble maps

The above technique called the 'Macbeath Trick' $\&$ is surprisingly useful, it can also be used to show that there are for example infinitely many soluble maps.

Let G be a soluble regular map of genus g which corresponds to a normal subgroup K of index n in $T = \triangle(2, m, k)$. Then again take the subgroup $L = [K, K] \cdot K^d$. Again we have that K/L is abelian (and therefore soluble) and T/L is a regular map.

We want to show that T/L is soluble. But this becomes clear once we apply the 2^{nd} group isomorphism theorem which tells us that

The same trick for soluble maps

The above technique called the 'Macbeath Trick' & is surprisingly useful, it can also be used to show that there are for example infinitely many soluble maps.

Let G be a soluble regular map of genus g which corresponds to a normal subgroup K of index n in $T = \triangle(2, m, k)$. Then again take the subgroup $L = [K, K] \cdot K^d$. Again we have that K/L is abelian (and therefore soluble) and T/L is a regular map.

We want to show that T/L is soluble. But this becomes clear once we apply the 2^{nd} group isomorphism theorem which tells us that

$$(T/L)/(K/L) \cong T/K \cong G$$

The same trick for soluble maps

The above technique called the 'Macbeath Trick' & is surprisingly useful, it can also be used to show that there are for example infinitely many soluble maps.

Let G be a soluble regular map of genus g which corresponds to a normal subgroup K of index n in $T = \triangle(2, m, k)$. Then again take the subgroup $L = [K, K] \cdot K^d$. Again we have that K/L is abelian (and therefore soluble) and T/L is a regular map.

We want to show that T/L is soluble. But this becomes clear once we apply the 2^{nd} group isomorphism theorem which tells us that

$$(T/L)/(K/L) \cong T/K \cong G$$

Therefore since both G and K/L are soluble then T/L must also be soluble and we have constructed another soluble quotient of $\triangle(2, m, k)$. Then simply vary d to obtain infinitely many soluble quotients.

Density of Regular Maps

The density of a set of positive integers $X \subseteq \mathbb{N}$ is defined as follows,

Let $X_n = X \cap \{1, 2, 3, \ldots, n\}$ then the density $\delta(X)$ of X is the limit

$$\delta(X) = \lim_{n \to \infty} \frac{|X|}{n}$$

if this limit does not exist we may replace it with either *limsup* or *liminf*.

Density of Regular Maps

Conjecture (Thomas Tucker)

Let $X_{m,k} \subseteq \mathbb{N}$ *be the set of genera of surfaces for which admit a regular map of type* $\{m, k\}$*. Then,*

$$\delta(X_{m,k}) = 0.$$

Or by abuse of notation $\delta(m, k)$*.*

Or equivalently, the density of the set of indexes of torsion free quotients of triangle groups is zero.

Progress so far

By a result of Bertram (1976) we can prove that for m, k, 2 relatively prime then $\delta(m,k) = 0$.

Theorem (Bertram's Theorem)

Let B be the set of numbers b such that any group G of order b has a normal cyclic sylow $p-$subgroup. Then $\delta(B) = 1$.

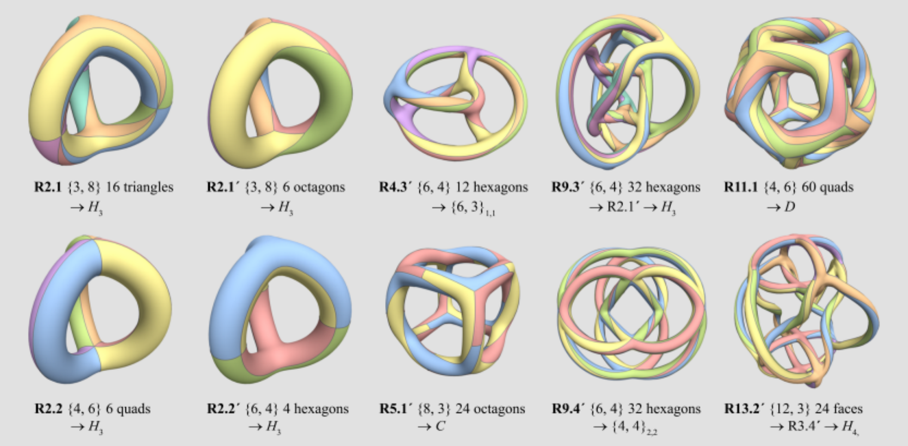
Using some reasonably elementary group theory we can show that no regular map with m, k and 2 relatively prime is a Bertram group and therefore the density of their orders and therefore by the Euler-Poincare formula the density of there genera is zero.

Density of soluble regular maps

Another density related conjecture (though this time we refer to density of certain types of maps within the set of regular maps as opposed to density of genus in the naturals) is about the density of soluble regular maps, it has been checked computationally by Marston Conder that about 95% of all orientably-regular maps of genus below 301 are soluble.

The reason for this is not at all obvious and though it is conjectured that this abundance of soluble maps continues for higher genus there is not much theoretical evidence to support this yet. Using the Macbeath trick we showed that if there exists one soluble map of type $\{m, k\}$ then there are infinitely many. It is a small part of my PhD project to prove that one soluble map exists for any hyperbolic map of type $\{m, k\}$ we have a method to construct soluble quotients that we believe will always work but the proof is not yet complete.

Thank you!


R2.1 {3, 8} 16 triangles
→ H_3
R2.1′ {3, 8} 6 octagons
→ H_3
R4.3′ {6, 4} 12 hexagons
→ $\{6, 3\}_{1,1}$
R9.3′ {6, 4} 32 hexagons
→ R2.1′ → H_3
R11.1 {4, 6} 60 quads
→ D
R2.2 {4, 6} 6 quads
→ H_3
R2.2′ {6, 4} 4 hexagons
→ H_3
R5.1′ {8, 3} 24 octagons
→ C
R9.4′ {6, 4} 32 hexagons
→ $\{4, 4\}_{2,2}$
R13.2′ {12, 3} 24 faces
→ R3.4′ → H_4