A Classification of Finite Groups with Three Automorphism Orbits

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Outline

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Backgrounds and Notations

Let N be a finite group and let $A = \mathrm{Aut}(N)$ be the full automorphism group of N.
The orbits of A acting on N are called *automorphism orbits* (*fusion classes*).

- $\omega(N)$: the number of automorphism orbits of N;
- $\pi(N)$: the set of orders of elements in N, called the *spectrum* of N.

Proposition

Elements in the same automorphism orbits have the same order. Hence $|\pi(N)| \leqslant \omega(N)$. In particular, it is well-known that

1. $\omega(N) = 1$ *if and only if* $N = 1$*;*
2. $\omega(N) = 2$ *if and only if* N *is an elementary abelian p-group.*

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Examples for N with $\omega(N) = 3$

The following groups has exactly 3 automorphism orbits:

1. $N = \mathbb{Z}_{p^2}^n$ for prime p with $\mathrm{Aut}(N) = \mathrm{GL}(n, \mathbb{Z}/p^2\mathbb{Z}) \cong p^{n^2}.\mathrm{GL}(n, p)$;
2. $N = Q_8$ with $\mathrm{Aut}(N) \cong \mathrm{S}_4$ and $\mathrm{Out}(N) \cong \mathrm{S}_3$;
3. $N = p_-^{1+2n}$ with $\mathrm{Out}(N) \cong \mathrm{CSp}(2n, p) = \mathrm{Sp}(2n, p){:}\mathbb{Z}_{p-1}$.

Example

Let p and q be two primes such that p is a primitive root modulo q (i.e. $q - 1$ is the least natural number e with $p^e \equiv 1 \mod q$). Then there exists a unique Frobenius group N isomorphic to $\mathbb{Z}_p^{n(q-1)}{:}\mathbb{Z}_q$ for positive integer n. Moreover, $\omega(N) = 3$.

The Frobenius groups defined above are exactly all (p, q)-groups N with $\omega(N) = 3$ (see [2, 6]).

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Another Example for N with $\omega(N) = 3$

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Let $q = p^n$ with prime p and positive integer n. Define

$$B_p(n) = \left\langle \begin{pmatrix} 1 & a & b \\ 0 & 1 & a^q \\ 0 & 0 & 1 \end{pmatrix} : b + b^q + aa^q = 0, \text{ for } a, b \in \mathbb{F}_{q^2} \right\rangle.$$

We remark that $B_p(n)$ is the upper-triangular unipotent subgroup of $\mathrm{SU}(3, q)$ on unitary space $\mathbb{F}_q^3$ equipped with the unitary form:

$$((x_1, x_2, x_3), (y_1, y_2, y_3)) = x_1 y_3 + x_2 y_2 + x_3 y_1.$$

In particular, $B_p(n)$ is a Sylow p-subgroup of $\mathrm{SU}(3, q)$.

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1. $|B_p(n)| = q^3 = p^{3n}$, and
2. $Z(B_p(n)) = \left\langle \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : b + b^q = 0, \text{ for } b \in \mathbb{F}_{q^2} \right\rangle \cong \mathbb{Z}_p^n.$

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Set $M(a,b) = \begin{pmatrix} 1 & a & b \\ 0 & 1 & a^q \\ 0 & 0 & 1 \end{pmatrix}$, and set $T = \begin{pmatrix} \lambda^{-q} & 0 & 0 \\ 0 & \lambda^{1-q} & 0 \\ 0 & 0 & \lambda \end{pmatrix}$, where λ is a generator of $\mathbb{F}_{q^2}^{\times}$. Then

1. $T^{-1}M(a,b)T = M(a\lambda, b\lambda^{q+1})$;
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Note that $|Z(B_p(n))| = q$ amd $|B_p(n)/Z(B_p(n))| = q^2$.

Lemma

For a generator λ of $\mathbb{F}_{q^2}^\times$, there exists an automorphism $\xi \in \mathrm{Aut}(B_p(n))$ such that $\xi(M(a, b)) = M(a\lambda, b\lambda^{q+1})$.

1. $\langle \xi \rangle$ *is transitive on* $Z(B_p(n))^*$;
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Some Observations

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Hence if $\omega(N) = 3$, then one of the followings holds:

1. N is a (p, q)-group (**Solved!**);
2. N is a p-group of exponent p^2;
3. N is a p-group of exponent p.

Observation 2: Suppose that $\omega(N) = 3$ and N is a p-group. Then N has exactly three characteristic subgroups:

$$\{\mathrm{id}_N\}, \quad \Phi(N) \ \text{ and } \ N.$$

Hence if N is a p-group of exponent p^2, then elements of the same order forms an automorphism orbit.

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A Primary Classification

Shult [8] proved a p-group for odd prime p, whose elements of order p forms an automorphism orbit, is abelian.

Corollary

Suppose that N is finite group with $\omega(N) = 3$. Then one of the followings holds:

1. *N is a Frobenius group of form $\mathbb{Z}_p^{n(q-1)}{:}\mathbb{Z}_q$;*
2. *$N \cong \mathbb{Z}_{p^2}^n$;*
3. *N is a non-abelian 2-group with exponent 4;*
4. *N is a non-abelian p-group with exponent p for odd prime p.*

We remark that if N satisfies case (3), then $\mathrm{Aut}(N)$ is transitive on the set of involutions of N.

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The definition of Suzuki 2-groups is given by Higman in [5].

Definition (Suzuki 2-group)

Let N be a non-abelian 2-group with more than one involution. If there exists a cyclic subgroup of $\mathrm{Aut}(N)$ which is transitive on involutions of N then N is called a *Suzuki 2-group*.

Higman proved that Suzuki 2-groups are one of four classes given in [5], named from A to D.

Proposition

Suppose that N is a Suzuki 2-group and $M = \langle x^2 \mid x \in N \rangle$. Then

1. *either N is of type A with $|M| = |N/M|$, or*
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The Suzuki 2-group of type A can be represent as (3×3)-matrix groups. Zhang [11] proved that Suzuki 2-group of type A has exactly 3 automorphism orbits.

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Suppose that N is a Suzuki 2-group of Type A with $|M| = |N/M| = 2^n$. Then there exists $1 \neq \theta \in \mathrm{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2)$ of odd order such that

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Example

Let ϵ is a multiplicative generator in $\mathbb{F}_{2^6}$ and let

$$N = P(\epsilon) = \{(a, x) \in \mathbb{F}_{2^6} \times \mathbb{F}_{2^3} \mid (a, x)(b, y) = (a + b, x + y + ab^2\epsilon + a^8 b^{16}\epsilon^8)\}.$$

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For any two generator ϵ_1 and ϵ_2, two groups $P(\epsilon_1)$ and $P(\epsilon_2)$ are isomorphic. Set $N = P(\epsilon_1)$ and $M = Z(N) \cong \mathbb{Z}_2^3$.

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Gross' Conjecture

Gross [4] extended the definition of Suzuki 2-groups.

Definition

If N is a 2-group with more than 1 involutions and all its involutions form an automorphism orbits, then N is called a 2-*automorphic* 2-*group*.

Then he proved the following theorem.

Theorem (Gross, 1967)

Suppose N is a 2-automorphic 2-group, then $M = \Omega_1(Z(N))$ contains all the involutions of N. One of the followings holds:

1. *N is homocyclic;*
2. *N has exponent 4 and nilpotent class 2, $|N| = |M|^2$ or $|M|^3$, where $M = N' = Z(N)$;*
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Theorem (Li & Zhu, 2022+)

Suppose that N is a 2-group with $M = Z(N) = N'$ and exponent 4 such that $|M| = |N/M|$ and involutions of N forms an automorphism orbits. Then $\omega(N) = 3$ and N is a Suzuki 2-group of type A.

Together with results of Wilkens and Bryukhanova, the Gross' conjecture is proved.

Theorem

Suppose that N is a non-abelian 2-group with at least 2 involutions and all involutions form an automorphism orbits. Then N is a Suzuki 2-group and $\mathrm{Aut}(N)$ is solvable.

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Classification on 2-groups

Corollary

Suppose that N is finite group with $\omega(N) = 3$. Then one of the followings holds:

1. *N is a Frobenius group of form $\mathbb{Z}_p^{n(q-1)}{:}\mathbb{Z}_q$;*
2. *$N \cong \mathbb{Z}_{p^2}^n$;*
3. *N is a non-abelian 2-group with exponent 4;*
4. *N is a non-abelian p-group with exponent p for odd prime p.*

The group N in case (3) is isomorphic to Q_8 or is a Suzuki 2-group, and $\mathrm{Aut}(N)$ is solvable. Finite group N with $\omega(N) = 3$ and $\mathrm{Aut}(N)$ solvable are known by Dornhoff [2]. Immediately, we obtain the following result.

Theorem (Li & Zhu, 2022+)

Suppose that N is a non-abelian 2-group with $\omega(N) = 3$. Then $N \cong A_2(n, \theta)$ with $1 \neq \theta$ of odd order, $B_2(n)$ or $P(\epsilon)$.

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An Extension of Extraspecial p-groups

Recall that extraspecial p-group p_-^{1+2n} of exponent p with odd prime p is the central product of Sylow p-subgroup of $\mathrm{SL}(3,p)$.

Example

Set Q the unipotent subgroup of $\mathrm{SL}(3,q)$ consists of upper-triangular matrices with 1's in the diagonal.
Define $N(m,q) = Q \circ Q \circ \cdots \circ Q$ be the central product of m copies of Q.

Lemma (Li & Zhu, 2022+)

Let $N = N(m,q)$ for $q = p^n$ with odd prime p, then $\omega(N) = 3$. Set $M = Z(N)$, then

$$\mathrm{Aut}(N)^{N/M} \cong \mathrm{C\Gamma Sp}(2m,q) \quad \textit{and} \quad \mathrm{Aut}(N)^{M} \cong \Gamma\mathrm{L}(1,q).$$

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3. *$N(m, q)/U$ for $q = p^n$ and odd prime p, where $U \leqslant N(m, q)'$ contains no subfield hyperplanes of $N(m, q)' \cong F^n$ for $F = \mathbb{F}_p$, and there exists $L < \Gamma\mathrm{L}(1, q)$ such that $U^g = U$ for each $g \in L$ and L is transitive on non-zero elements of $N(m, q)'/U$.*

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3. $A_2(n, \theta)$, *the Suzuki 2-group of type A;*
4. $B_2(n)$, *a Sylow 2-subgroup of* $\mathrm{SU}(3, 2^n)$;
5. $P(\epsilon)$, *a group of order* 2^9;
6. $A_p(n, \theta)$ *with odd prime* p *such that* $3 \mid n$ *and* $|\theta| = 3$;
7. $N(m, q) = B_p(n) \circ \cdots \circ B_p(n)$, *the central product of* m *copies of Sylow* p*-subgroup of* $\mathrm{SU}(3, q)$, *with* $q = p^n$ *and* p *an odd prime;*
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Outline

UCS p-groups

Suppose that N is a non-abelian p-group with $\omega(N) = 3$.

- Then N has a **u**nique non-trivial proper **c**haracteristic **s**ubgroup $M = Z(N) = N' = \Phi(N)$.
- Taunt [9] named such groups by *UCS groups*.
- The UCS p-groups are studied by Glasby and Pálfy and Schneider in [3], and they provide a tool in representation theory to study such groups.

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Recall the *exterior square* of FG-module V:

$$\Lambda_F^2(V) = (V \otimes_F V)/S \text{ where } S = \langle v \otimes v \mid v \in V \rangle.$$

Lemma

Let $G = \mathrm{Aut}(N)^{N/M}$ and let $F = \mathbb{F}_p$. Then both M and N/M are FG-modules, and M is isomorphic to a quotient FG-module of $\Lambda_F^2(N/M)$.

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Transitive Modules with Transitive Quotient of its Exterior Square

We say V is a *transitive* FG-module if G acts transitively on non-zero vectors of V.

Theorem (Li & Zhu, 2022+)

Suppose that V is a faithful transitive FG-module for $F = \mathbb{F}_p$ where G is nonsolvable. If there exists a transitive quotient FG-module W of $\Lambda^2_F(V)$, then one of the followings holds:

1. $\dim_F W = 1$;
2. *G contains a normal subgroup $G_0 = \mathrm{SL}(3, q)$ for $q = p^n$ where $\dim_F V = \dim_F W = 3n$ and W is the dual FG_0-module of V;*
3. *G contains a normal subgroup $G_0 = \mathrm{Sp}(2m, q)$ or $\mathrm{G}_2(q)$ for $q = p^n$ such that W is a trivial FG_0-module and is a faithful transitive quotient FH-module of F^n where $H = G/G_0 \lesssim \Gamma\mathrm{L}(1, q)$.*

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Theorem (Li & Zhu, 2022+)

Suppose that V is a faithful transitive FG-module for $F = \mathbb{F}_p$ where G is nonsolvable. If there exists a transitive quotient FG-module W of $\Lambda_F^2(V)$, then one of the followings holds:

1. $\dim_F W = 1$;
2. *G contains a normal subgroup $G_0 = \mathrm{SL}(3, q)$ for $q = p^n$ where $\dim_F V = \dim_F W = 3n$ and W is the dual FG_0-module of V;*
3. *G contains a normal subgroup $G_0 = \mathrm{Sp}(2m, q)$ or $\mathrm{G}_2(q)$ for $q = p^n$ such that W is a trivial FG_0-module and is a faithful transitive quotient FH-module of F^n where $H = G/G_0 \lesssim \Gamma\mathrm{L}(1, q)$.*

Outline

Sketch of Proof for Gross' Conjecture

Suppose that N is a non-abelian 2-automorphic 2-group with $|N'| = |N/N'|$. Set $M = N'$ and set $G = \mathrm{Aut}(N)^{N/M}$.

- If G is solvable, then N is a Suzuki 2-group by Shaw's Theorem [7].
- If G is nonsolvable, then $\omega(N) = 3$ with $|M| = |N/M|$.
  Hence $G_0 \cong \mathrm{SL}(3, q) \lhd G$ and N/M is FG_0-isomorphic to the dual FG_0-module of M.
  Let $v \in N \setminus M$ and let $\overline{v}$ be the image of v in N/M, then $(G_0)_{\overline{v}} = (G_0)_{v^2}$.
  Thus M cannot be the dual FG_0-module of N/M, a contradiction.

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Suppose that N is a finite group with $\omega(N) = 3$.

- If N is not a p-group, then N is a (p, q)-group. In this case, N is a Frobenius group of form $\mathbb{Z}_p^{n(q-1)}{:}\mathbb{Z}_q$.
- Suppose that N is an abelian p-group. Then $\Omega_1(N) = \mho_1(N)$, and hence $N = \mathbb{Z}_{p^2}^n$.
- Suppose that N is a non-abelian 2-group. Then N is a Suzuki 2-group with $\mathrm{Aut}(N)$ solvable. With classification given by Dornhoff [2], N is isomorphic to one of the followings:
  1. $N = A_2(n, \theta)$ for $1 \neq \theta$ of odd order;
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Suppose that N is a non-abelian p-group for odd prime p and $\omega(N) = 3$. Set $M = N'$ and set $G = \mathrm{Aut}(N)^{N/M}$.

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Subfield Hyperplane

Suppose that $H = \langle x \rangle{:}\langle y \rangle \cong \Gamma\mathrm{L}(1, p^n)$ acts naturally on $W = \mathbb{F}_p^n$.

Definition (Subfield Hyperplane)

An $(n-d)$-dimensional subspace U of W is called a *subfield hyperplane* if $d \mid n$ and $W^{x^\ell} = W$ for $\ell = \frac{p^n-1}{p^d-1}$.

Example

Let $W = \mathbb{F}_{3^4}^+$ and let λ be a generator of $\mathbb{F}_{3^4}^\times$ with minimal polynomial $\lambda^4 - \lambda^3 - 1 = 0$. Set $H = \langle x \rangle{:}\langle y \rangle \cong \Gamma\mathrm{L}(1, 3^4)$ with x admitted by multiplying λ and y admitted by Frobenius automorphism $a \mapsto a^3$. Let $L = \langle xy \rangle$, then $W = U_1 \oplus U_2$ with transitive $\mathbb{F}_3 L$-submodules

$$U_1 = \langle 1+\lambda^2,\ 2\lambda+\lambda^2+2\lambda^3 \rangle \text{ and } U_2 = \langle 2+\lambda,\ 1+\lambda+\lambda^2 \rangle.$$

Neither of them is a subfield hyperplane.

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Case 8 in the Classification

We remark that the center (commutator subgroup) of a Sylow p-subgroup of $\mathrm{SL}(3, p^n)$ is isomorphic to $\mathbb{F}_{p^n}^+$. Thus

$$N(m,q)' = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : b \in \mathbb{F}_{p^n} \right\} \cong \mathbb{F}_p^+.$$

Then $N(m,q)'$ is a natural $\mathbb{F}_p\Gamma\mathrm{L}(1,p^n)$-module.

Case 8

$N(m,q)/U$ for $q = p^n$ and odd prime p, where $U \leqslant N(m,q)'$ contains no subfield hyperplanes of $N(m,q)' \cong F^n$ for $F = \mathbb{F}_p$, and there exists $L < \Gamma\mathrm{L}(1,q)$ such that $U^g = U$ for each $g \in L$ and L is transitive on non-zero elements of $N(m,q)'/U$.

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An example of Case 8

Example

Let $N = N(1, 3^4)/U$ where $N(1, 3^4)$ is the upper-triangular unipotent subgroup of $\mathrm{SL}(3, 3^4)$ and

$$U = \left\langle \begin{pmatrix} 1 & 0 & 1+\lambda^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2\lambda+\lambda^2+2\lambda^3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle,$$

where λ is a generator of $\mathbb{F}_{3^4}^{\times}$ with minimal polynomial $\lambda^4 - \lambda^3 - 1 = 0$. Then $|N| = 3^{10}$, $\omega(N) = 3$ and $\mathrm{Aut}(N)^{N/N'} \cong \mathrm{Sp}(2, 3^4){:}\mathbb{Z}_8$

Outline

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Thanks!