Random Walks on Vertex-Transitive Graphs with Moderate Growth

(A generalisation of a result by P. Diaconis and L. Saloff-Coste)

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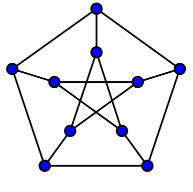
Definition

A graph Γ is called vertex-transitive if for any two vertices $u, v \in \Gamma$, there is $f \in \mathrm{Aut}(\Gamma)$ such that $f(u) = f(v)$, i.e. $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$.

Fact

Every Cayley graph is vertex-transitive, but there exist vertex-transitive graphs that are not Cayley graphs.

Example (Petersen Graph)



Definitions and Examples

Notation

Given a vertex transitive graph Γ, we denote $\gamma = \operatorname{diam}(\Gamma)$ to be the diameter of the graph Γ, and $\beta(j)$ to be the size of the ball of radius j in Γ.

Definition

Given constants A, d, we say a finite vertex-transitive graph Γ has (A, d)-moderate growth if

$$\beta(j) \geq \frac{1}{A}\left(\frac{j}{\operatorname{diam}(\Gamma)}\right)^d |\Gamma| \quad \textit{whenever} \quad 1 \leq j \leq \operatorname{diam}(\Gamma).$$

Example (example of classes of Cayley graphs with moderate growths)

- *For $m \in \mathbb{N}$, $\operatorname{Cay}(Z_m, \{-1, 0, 1\})$ has $(1, 1)$-moderate growth.*
- *The Cayley graphs for finite nilpotent groups have (A, d)-moderate growth, where A and d dependent only on the number of generators and the class of nilpotency.*
- *The Cayley graphs for the finite affine groups over $\mathbb{F}_p$ have $(1, 2)$-moderate growth*

Simple Random Walk on Cayley graphs

Definition

Let G be a finite group with generating set $S \ni 1$. The probability vector q associated with simple random walk *on $\Gamma = \mathrm{Cay}(G, S)$ is defined as follows:*

$$q(s) = \begin{cases} \frac{1}{|S|} & \text{if } s \in S, \\ 0 & \text{otherwise.} \end{cases}$$

The probability vector u associated with uniform distribution *on Γ is defined as $u(g) = \frac{1}{|G|}$, for all $g \in G$.*

Theorem (Quadratic mixing time on Cayley graph, P. Diaconis & L. Saloff-Coste, 1994)

Suppose $\mathrm{Cay}(G, S)$ has (A, d)-moderate growth with respect to S. Then, for $n = (1 + c)|S|\gamma^2, c > 0$, we have

$$\left\| q^{(n)} - u \right\|_{T.V} < Be^{-c}$$

with $B = A^{1/2} 2^{d(d+3)/4}$.

Random Walk on Vertex-transitive graphs

Recall

- *A transition system* (Γ, P) *consist of a graph* Γ *and a transition matrix* P*.*
- *For* $u, v \in \Gamma$*, we denote by* P_u *the* u*-th row of* P*, this is a probability vector with non-negative entries that add up to* 1*; the* (u, v)*th entry of* P*, denoted by* $P(u, v)$*, tells us the probability of going from* u *to* v*.*
- *We denote by* U *the matrix associated with uniform distribution, i.e.* $U(u, v) = \frac{1}{|\Gamma|}$ *for all* $u, v \in V$*.*

Definition (transitive system)

We define the set of automorphisms on Γ *preserved by* P *to be*

$$Aut(\Gamma, P) = \{\alpha \in \mathrm{Aut}(\Gamma) : \forall u, v \in V, P(u, v) = P(\alpha(u), \alpha(v))\}.$$

We say that the system (Γ, P) *is transitive if* $\mathrm{Aut}(\Gamma, P)$ *acts transitively on* $V(\Gamma)$*.*

Lemma

Suppose (Γ, P) *is a transitive system, then for* u*,* $v \in \Gamma$*, there exist* $\alpha \in \mathrm{Aut}(\Gamma)$*, such that* $P(u, w) = P(v, \alpha(w))$ *for all* $w \in \Gamma$*.*

Mixing time on VTG with moderate growth

Definition

For $\epsilon > 0$ and $1 \leq p \leq \infty$, the ℓ^p mixing time of system (Γ, P) is

$$\tau_p(\epsilon) = \min\{n \geq 0 : \|(P^n - U)_o\|_p \leq \epsilon \|U_o\|_p\}$$

Theorem (Moderate growth implies quadratic mixing time)

Let (Γ, P) be a symmetric transitive system that undergoes (A, d)- moderate growth.
Suppose that $\eta = \inf\{P(u, v) : u \sim v\} > 0$ and $\delta = P(o, o) \geq \frac{\eta}{2\gamma^2}$.
Then the mixing time for the system (Γ, P) is bounded above by

$$\tau_p(\epsilon) \leq \frac{\gamma^2}{\eta} \log \frac{A 2^{2+\frac{d+d^2}{2}} e^2}{\epsilon}.$$

Main idea of the proof

Lemma

- *The mixing time* $\tau_p(\epsilon)$ *is non-decreasing in* p.
- *We have* $\tau_\infty(\epsilon) = 2\tau_2(\sqrt{\epsilon})$.

Lemma (Bound for ℓ^2 mixing time)

Suppose (Γ, P) *is a symmetric system. Let* $1 = \pi_1 \geq \pi_2 \geq \ldots \geq \pi_N \geq -1$ *be the eigenvalues of* P *and define* $\pi_* = \max\{|\pi_2|, |\pi_N|\}$. *Then,*

- *For positive* n *and* m, $\|(P^{n+m} - U)_o\|_2^2 \leq P^{2m}(o,o)\pi_*^{2n}$.
- *We have* $\pi_N \geq -1 + 2\inf\{P(u,u) | u \in V\}$.
- *If* Γ *is a vertex-transitive graph,* $\pi_2 \leq 1 - \inf\{P(u,v) : u \sim v\}/\gamma^2$.

Theorem (Bound for return probability)

Let (Γ, P) *be a symmetric transitive system with* (A, d) *- moderate growth. Suppose that* $\eta = \inf\{P(u,v) : u \sim v\} > 0$. *Then, for* $m = \frac{4\gamma^2}{\eta}$,

$$P^{2m}(o,o) \leq 2^{2+\frac{d+d^2}{2}} A/|\Gamma|.$$

Theorem (Large diameter implies moderate growth, R. Tessera & M. Tointon 2021)

Let Γ *be a finite connected vertex-transitive graph. For every* $\delta \geq 0$*, there exists* $n_0 = n_0(\delta)$ *such that if* $\operatorname{diam}(\Gamma) \geq n_0$ *and*

$$\operatorname{diam}(\Gamma) \geq \left(\frac{|\Gamma|}{\beta(1)}\right)^{\delta} \tag{1}$$

then Γ *has* $(\mathcal{O}_\delta(1), \mathcal{O}_\delta(1))$ *moderate growth.*

Corollary (Large diameter implies quadratic mixing time)

Let $\delta \geq 0$*. Suppose our system* (Γ, P) *also satisfies the large diameter condition (1), then* (Γ, P) *has quadratic mixing time, i.e. for* $1 \leq p \leq \infty$*,*

$$\tau_p(\epsilon) = \mathcal{O}_\delta\left(\frac{\gamma^2}{\eta}\log \mathcal{O}_\delta\left(\frac{1}{\epsilon}\right)\right).$$

Thank you for listening!