On CI-property of normal circulant digraphs

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Outline

1 Introduction

Cayley digraph

Let G be a finite group and let S be a subset of G with $1 \notin S$. The *Cayley digraph* $\mathrm{Cay}(G, S)$ of G with respect to S is defined to be the digraph with vertex set G and arc set $\{(g, sg) \mid g \in G, s \in S\}$.

Cayley digraphs isomorphism problem

Given two Cayley digraphs $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, T)$. Decide whether they are isomorphic.

Cayley isomorphism

Cayley digraphs $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, T)$ are called *Cayley isomorphic*, if $\exists \sigma \in \mathrm{Aut}\,(G)$ such that $S^\sigma = T$.

1 Introduction

CI-subset, or CI-(di)graph

A Cayley subset S of G is called a *CI-subset* if for any $T \subseteq G$, $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$ implies that $T = S^{\varphi}$ for some $\varphi \in \mathrm{Aut}\,(G)$, and in this case, we call $\mathrm{Cay}(G, S)$ a *CI-(di)graph*.

Cayley isomorphism problem

Given two subsets $S, T \subseteq H$. Find whether they are conjugate by an element of $\mathrm{Aut}\,(G)$.

(D)CI-group

A group G is said to be a *DCI-group* if every Cayley digraph of G is a CI-digraph and it is said to be a *CI-group* if every Cayley graph of G is a CI-graph. Thus, every DCI-group is a CI-group. Every subgroup of a (D)CI-group is also a (D)CI-group.

1 Introduction

- Investigation of the isomorphism problem of Cayley graphs started with Ádám's conjecture [1]: *every cyclic group is a (D)CI-group.*
- Disproved by Elspas and Turner [2] for directed Cayley graphs of $\mathbb{Z}_8$ and for undirected Cayley graphs of $\mathbb{Z}_{16}$.
- Muzychuk [3,4] proved that *a cyclic group $\mathbb{Z}_n$ is a DCI-group if and only if $n = k$ or $n = 2k$, where k is square-free.* Furthermore, *$\mathbb{Z}_n$ is a CI-group if and only if n is as above or $n = 8, 9, 18$.*
- Babai and Frankl [5] also asked *whether every elementary abelian p-group is a (D)CI-group.*

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[1] A. Ádám, Research Problem 2 – 10, J. Combin. Theory 2 (1967), 393.

[2] B. Elspas, J. Turner, Graphs with circulant adjacency matrices, J. Combin. Theory, 9 (1970), 297 – 307.

[3] M. Muzychuk, *Ádám*'s conjecture is true in the square-free case, J. Combin. Theory Ser. A 72 (1995), 118 – 134.

[4] M. Muzychuk, On *Ádám* s conjecture for circulant graphs, Discrete Math. 167/168 (1997), 497 – 510; corrigendum 176 (1997), 285 – 298.

[5] L. Babai, P. Frankl, Isomorphisms of Cayley graphs I, in: Colloqzeria Mathematica Societatis János Bolyai, Vol. 18. Combinatorics, Keszthely, 1976, North-Holland, Amsterdam (1978), 35 – 52.

1 Introduction

- C_p [6, D. Ź. Djoković; 7, B. Elspas, J. Turner]
- C_p^2 [8, C.D. Godsil]
- C_p^3 [9, E. Dobson]
- C_p^4 [10, J. Morris]
- C_p^5 [11, Y-Q. Feng, I. Kovács]
- C_p^r is not a CI-group if $r \geq 2p + 3$ [12, G. Somlai]

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[6] D. Ź. Djoković, Isomorphism problem for a special class of graphs, Acta Math. Acad. Sci. Hungar. 21 (1970), 267 – 270.

[7] B. Elspas, J. Turner, Graphs with circulant adjacency matrices, J. Combin. Theory, 9 (1970), 297 – 307.

[8] C.D. Godsil, On Cayley graph isomorphisms, Ars Combin. 15 (1983) 231 – 246.

[9] E. Dobson, Isomorphism problem for Cayley graphs of $\mathbb{Z}_p^3$, Discrete Math. 147 (1995), 87 – 94.

[10] J. Morris, Elementary proof that $\mathbb{Z}_p^4$ is a DCI-group, Discrete Math. 338 (2015), 1385 – 1393.

[11] Y-Q. Feng, I. Kovács, Elementary abelian groups of rank 5 are DCI-groups, J. Combin. Theory. Series A. 157 (2018), 162 – 204.

[12] G. Somlai, Elementary abelian p-groups of rank $2p + 3$ are not CI-groups, Journal of Algebraic Combinatorics, 34(3) (2011), 323 – 335.

1 Introduction

- $C_p^2 \times C_q$ [13, I. Kovács, M. Muzychuk]
- $C_p^3 \times C_q$ [14, M. Muzychuk, G. Somlai]
- $C_p^4 \times C_q$ [15, I. Kovács, G. Ryabov]
- $C_2^5 \times C_p$ [16, G. Ryabov]
- $C_p^2 \times C_4$ [17, G. Ryabov]
- $C_p \times C_n$ (n square-free), $C_p^2 \times C_q^2$ [18, I. Kovács, M. Muzychuk, P. P. Pálfy, G. Ryabov, G. Somlai]

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[13] I. Kovács, M. Muzychuk, The group $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ is a CI-group, Comm. Algebra 37 (2009) 3500-3515.

[14] M. Muzychuk, G. Somlai, The Cayley isomorphism property for $\mathbb{Z}_p^3 \times \mathbb{Z}_q$, Algebraic Combinatorics 4 (2021) 289-299.

[15] I. Kovács, G. Ryabov, The group $C_p^4 \times C_q$ is a DCI-group, Discrete Math. 345 (2022) 112705.

[16] G. Ryabov, The Cayley isomorphism property for the group $C_2^5 \times C_p$, ArXiv:2005.14539v1

[17] G. Ryabov, The Cayley isomorphism property for the group $C_4 \times C_p^2$, Comm. Algebra 49 (2021), 1788 C1804.

[18] I. Kovács, M. Muzychuk, P. P. Pálfy, G. Ryabov, G. Somlai, CI-property of $C_p^2 \times C_n$ and $C_p^2 \times C_q^2$ for digraphs, arXiv:2201.02725v1.

1 Introduction

$E(M, n)$

Let M be an abelian group for which all Sylow subgroups are elementary abelian. Let $n \in \{2, 3, 4, 8, 9\}$ such that $(|M|, n) = 1$. Let $E(M, n) = M \rtimes \langle z \rangle$ such that $o(z) = n$.

- If $o(z) = n$ is even, $x^z = x^{-1}$ for all $x \in M$.
- If $o(z) = 3$ or 9, $x^z = x^l$ for all $x \in M$, $l^3 \equiv 1 (\mod \exp(M))$ and $(l(l-1), \exp(M)) = 1$.

[19] Cai Heng Li, On isomorphisms of finite Cayley graphs–a survey, Discrete Math. 256 (2002), 301-334.

A more precise list of candidates for CI-groups was given by Li, Lu and Pálfy [20].

Theorem 1.2.

Let G be a finite CI-group.

(a) If G does not contain elements of order 8 or 9, then $G = H_1 \times H_2 \times H_3$, where the orders of H_1, H_2, and H_3 are pairwise coprime, and
  (i) H_1 is an abelian group, and each Sylow p-subgroup of H_1 is elementary abelian or $\mathbb{Z}_4$;
  (ii) H_2 is one of the groups $E(M, 2)$, $E(M, 4)$, $\mathbb{Q}_8$,or 1;
  (iii) H_3 is one of the groups $E(M, 3)$, A_4, or 1.

(b) If G contains elements of order 8, then $G \cong E(M, 8)$ or $\mathbb{Z}_8$.

(c) If G contains elements of order 9, then G is one of the groups $\mathbb{Z}_9 \rtimes \mathbb{Z}_2$, $\mathbb{Z}_9 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_9$, or $\mathbb{Z}_9 \times \mathbb{Z}_2^n$ with $n \leq 5$.

[20] C. H. Li, Z. P. Lu, P. P. Pálfy, Further restrictions on the structure of finite CI-groups, J. Algebr. Comb. 26 (2007), 161 − 181.

The known DCI- or CI-groups are:

- $\mathbb{Z}_n$, where $n \in \{3, 9, 18, k, 2k, 4k\}$ and k is odd and square-free [15, 16];
- $\mathbb{Z}_p^2$ [10]; $\mathbb{Z}_p^3$ [5]; $\mathbb{Z}_p^4$ ([19] for $p = 2$, [11] for $p > 2$, and [14] independently);
- D_{2p} [1];
- F_{3p} (the Frobenius group of order $3p$) [6, Theorem 21], see also [7] and [13];
- $\langle a, z \mid a^p = 1, z^r = 1, z^{-1}az = a^{-1} \rangle$ where r is 4 or 8 [13];
- $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ (and $\mathbb{Z}_2^2 \times \mathbb{Z}_p$) [9];
- $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ [12];
- $Q_8 \times \mathbb{Z}_p$ [20];
- some specific small groups: Q_{12}, A_4, $\mathbb{Z}_3^2.\mathbb{Z}_2$, $\mathbb{Z}_3.\mathbb{Z}_8$, Q_{28}, $D_{10} \times \mathbb{Z}_3$, $D_6 \times \mathbb{Z}_5$, D_{30}, and all of their subgroups [19]; $\mathbb{Z}_2^5$ [4]; and $\mathbb{Z}_3^5$ [21]; and
- $D_{2n}, \mathbb{Z}_n \rtimes \mathbb{Z}_3, \mathbb{Z}_p^2 \times \mathbb{Z}_n$ [7], and $\mathbb{Z}_p^2 \times \mathbb{Z}_q \times \mathbb{Z}_n$ [8], where n satisfies $\gcd(n, \varphi(n)) = 1$ (φ is Euler's phi function) and in the first three cases other additional arithmetic conditions.

Theorem 1 *Let p be a prime number, and let R be the dihedral group of order $6p$. Then, R is a* DCI*-group if and only if $p \geq 5$, and R is a* CI*-group if and only if $p \geq 3$.*

[21] E. Dobson, J. Morris, P. Spiga, Further restrictions on the structure of finite DCI-groups: an addendum, J. Algebr. Comb. 42(2015), 959 – 969.

- Nomal Cayley (di)graph: If the right regular representation $R(G)$ of G is a normal subgroup of $\mathrm{Aut}\,(\mathrm{Cay}(G,S))$.
- CI-(di)graph: $\Gamma = \mathrm{Cay}(G,S)$ is said to a CI-(di)graph if for every $T \subseteq G$, $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$ implies that $T = S^{\sigma}$ for some $\sigma \in \mathrm{Aut}\,(G)$.

[19, Example 6.10]

Let $G = \langle a \rangle \cong C_{2^r}$ where $r \geq 3$. Set $S = \{a, a^{2^{r-1}+1}, a^2\}$, and let $\Gamma = \mathrm{Cay}(G,S)$. Then Γ is a normal Cayley digraph but not a CI-digraph.

[19, Problem 6.11]

Characterize normal Cayley (di)graphs which are not CI-(di)graphs.

[19] C. H. Li, On isomorphisms of finite Cayley graphs–a survey, Discrete Math. 256 (2002), 301-334.

NDCI-group and NCI-group

- **NDCI-group:** The group G is called a NDCI-group if all normal Cayley digraphs of G are CI-graphs.

- **NCI-group:** The group G is called a NCI-group if all normal Cayley graphs of G are CI-graphs.

Problem 1.1

Classify finite NDCI-groups and NCI-groups.

[0] J.-H. Xie, Y.-Q. Feng, G. Ryabov, Y.-L. Liu, Normal Cayley digraphs of cyclic groups with CI-property, Comm. Algebra 50 (2022), 2911-2920.

Main results

The Cyclic-NDCI-groups and NCI-groups.

Theorem 1.2

A cyclic group C_n of order n is a NDCI-group if and only if $8 \nmid n$.

Corollary 1.3

A cyclic group C_n of order n is a NCI-group if and only if either $n = 8$ or $8 \nmid n$.

[0] J.-H. Xie, Y.-Q. Feng, G. Ryabov, Y.-L. Liu, Normal Cayley digraphs of cyclic groups with CI-property, Comm. Algebra 50 (2022), 2911-2920.

2 Preliminaries

Proposition 2.1 [22, Propositions 1.3 and 1.5]

Let $\mathrm{Cay}(G, S)$ be a Cayley digraph of a group G with respect to S and let $A = \mathrm{Aut}\,(\mathrm{Cay}(G, S))$. Then $N_A(R(G)) = R(G) \rtimes \mathrm{Aut}\,(G, S)$ and $\mathrm{Cay}(G, S)$ is normal if and only if $A_1 = \mathrm{Aut}\,(G, S)$.

Proposition 2.2 [19, Corollary 6.9]

Let $\mathrm{Cay}(G, S)$ be a normal Cayley digraph of a group G with respect to S. Then $\mathrm{Cay}(G, S)$ is a CI-digraph if and only if $R(G)$ is the unique regular subgroup of $\mathrm{Aut}\,(\mathrm{Cay}(G, S))$ which is isomorphic to G.

Proposition 2.3 [19, Example 6.10]

Let $G = \langle a \rangle$ with $o(a) = 2^r \geq 8$ and let $S = \{a, a^2, a^{2^{r-1}+1}\}$. Then $\mathrm{Cay}(G, S)$ is a normal Cayley digraph but not a CI-digraph.

[19] C. H. Li, On isomorphisms of finite Cayley graphs–a survey, Discrete Math.

[22] M. Y. Xu, Automorphism groups and isomorphisms of Cayley digraphs, Discrete Math.

Proposition 2.4 [Muzychuck or [19, Theorem 7.1]]

A cyclic group of order n is a DCI-group if and only if $n = k, 2k$ or $4k$ where k is odd square-free, and a cyclic group of order n is a CI-group if and only if either $n \in \{8, 9, 18\}$ or $n = k, 2k$ or $4k$ where k is odd square-free.

Proposition 2.5 [23, Theorem 7.3]

For a positive integer n, let $n = \prod_{i=1}^{m} p_i^{r_i}$ be the distinct prime factorization of n, and let $C_n = C_{p_1^{r_1}} \times C_{p_2^{r_2}} \times \cdots \times C_{p_m^{r_m}}$. Then $\mathrm{Aut}\,(C_n) = \mathrm{Aut}\,(C_{p_1^{r_1}}) \times \mathrm{Aut}\,(C_{p_2^{r_2}}) \times \cdots \times \mathrm{Aut}\,(C_{p_m^{r_m}})$. Furthermore, $\mathrm{Aut}\,(C_2) = 1$, $\mathrm{Aut}\,(C_4) \cong C_2$, $\mathrm{Aut}\,(C_{2^n}) \cong C_2 \times C_{2^{n-2}}$ for $n \geq 2$, and $\mathrm{Aut}\,(C_{p^n}) \cong C_{(p-1)p^{n-1}}$ for an odd prime p.

[19] C. H. Li, On isomorphisms of finite Cayley graphs–a survey, Discrete Math. 256 (2002), 301-334.

[23] J. J. Rotman, An Introduction to the Theory of Groups, Fourth Edition. Springer-Verlag. 1995.

Proposition 2.6

Let G be a finite group, and let $1 \notin S \subset G$, $1 < H \leq K < G$ and $H \trianglelefteq G$. Assume that $S \backslash K$ is a union of some cosets of H in G and that there exist $x \notin K$ and $y \in H$ such that $y^x \neq y^{-1}$. Then the Cayley digraph $\mathrm{Cay}(G, S)$ is non-normal.

- Take $u, v \in G$ with $Ku \neq Kv$. $[Hk_1u, Hk_2v]$ is isomorphic to either the empty graph, or $\vec{K}_{|H|,|H|}$. If an automorphism of $[Ku]$ of Ku in Γ fixes every coset of H in Ku setwise, then it can be extended to an automorphism of Γ fixing $G \backslash Ku$ pointwise.
- For $\forall h \in H$, define $\bar{h}$ to be the permutation on $V(\Gamma)$ such that $\bar{h}$ fixes $G \backslash Kx$ pointwise and $t^{\bar{h}} = t^{R(h)} = th$ for $\forall t \in Kx$. Then we have $\bar{H} := \{\bar{h} \mid h \in H\} \leq A_1$.
- Suppose to the contrary that Γ is normal. $\bar{H}$ fixes Kx^2 pointwise. And we have $(x^2)^{\bar{y}} = x^2$ and $x^{\bar{y}} = xy$ as $x \in Kx$. However, $(x^2)^{\bar{y}} = x^{\bar{y}} x^{\bar{y}} = xyxy$. Then $x^{-1}yx = y^{-1}$, a contradiction.

Proof of Theorem 1.1

Theorem 1.1

(i) A cyclic group C_n of order n is a NDCI-group if and only if $8 \nmid n$.

(ii) A cyclic group C_n of order n is a NCI-group if and only if either $n = 8$ or $8 \nmid n$.

Proof of Theorem 1.1

Lemma 3.1

For a positive integer n, let $n = \prod_{i=1}^{m} p_i^{r_i}$ be the distinct prime factorization of n, and let $\mathrm{Cay}(C_n, S)$ be a Cayley digraph of the cyclic group $C_n = C_{p_1^{r_1}} \times C_{p_2^{r_2}} \times \cdots \times C_{p_m^{r_m}}$. Assume that p_t is an odd prime and that $\mathrm{Aut}\,(C_n, S)$ contains an element of order p_t in $\mathrm{Aut}\,(C_{p_t^{r_t}})$ for some $1 \le t \le m$. Then $\mathrm{Cay}(C_n, S)$ is non-normal.

- $(a_1^{j_1} \ldots a_{t-1}^{j_{t-1}} a_t^{j_t} a_{t+1}^{j_{t+1}} \ldots a_m^{j_m})^{\delta} = a_1^{j_1} \ldots a_{t-1}^{j_{t-1}} (a_t^{j_t})^{p_t^{r_t-1}+1} a_{t+1}^{j_{t+1}} \ldots a_m^{j_m}$. Then $\langle\delta\rangle \le \mathrm{Aut}\,(C_n, S)$. ($\delta$: $a_t \mapsto a_t^{p_r^{r_t-1}+1}$.)
- Let $K = \langle a_1, a_2, \cdots, a_{t-1}, a_t^{p_t}, a_{t+1}, \cdots, a_m\rangle$, and $H = \langle a_t^{p_t^{r_t-1}}\rangle$. By taking $x = a_t$ and $y = a_t^{p_t^{r_t-1}}$, we have $y^x \ne y^{-1}$.
- Take $\forall z \in S\backslash K$. Then z_t has order $p_t^{r_t}$ as $C_n = \prod_{i=0}^{p_t-1} a_t^i K$, implying $z_t^{p_t^{r_t-1}} \ne 1$. Since $z^{\delta} = z(z_t)^{p_t^{r_t-1}}$, $\langle\delta\rangle$ is transitive on zH, and since $\langle\delta\rangle \le \mathrm{Aut}\,(C_n, S) \le A$, we have $zH \subseteq S\backslash K$.

Lemma 3.2

Let m and s be positive integers such that $s \geq 3$, $(2, m) = 1$ and $2^s m \neq 8$. Let $C_{2^s m} = \langle a \rangle \times \langle b \rangle$ with $o(a) = 2^s$ and $o(b) = m$, and $S = \{(ab)^{\pm 1}, (a^2b^2)^{\pm 1}, (a^{2^{s-1}+1}b)^{\pm 1}\}$. Then $\mathrm{Cay}(C_{2^s m}, S)$ is a normal Cayley graph but not a CI-graph.

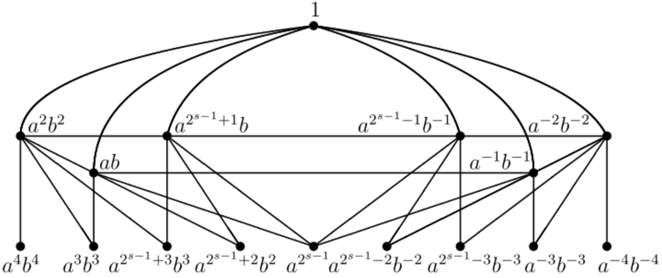

FIGURE 1. The induced subgroup $[\Gamma_0(1) \cup \Gamma_1(1) \cup \Gamma_2(1)]$ in Γ.

Proof of Theorem 1.1

A cyclic group C_n of order n is a NDCI-group if and only if $8 \nmid n$.

- Let $8 \mid n$. By Proposition 2.3 and Lemma 3.2, there exists a normal non-CI Cayley digraph on C_n. The necessity follows.
- To prove the sufficiency, assume that $8 \nmid n$ and we only need to prove that C_n is a NDCI-group.
- For convenience, write $n = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m} 2^s$ and we may assume that $m \geq 1$. Let $\Gamma = \mathrm{Cay}(C_n, S)$ be a normal Cayley digraph and let $A = \mathrm{Aut}\,(\Gamma)$. To prove that C_n is a NDCI-group, it suffices to show that Γ is a CI-digraph.

Proof of Theorem 1.1

- Let $C_n = C_{p_1^{r_1}} \times C_{p_2^{r_2}} \times \cdots C_{p_m^{r_m}} \times C_{2^s}$. Then $\mathrm{Aut}\,(C_n) = \mathrm{Aut}\,(C_{p_1^{r_1}}) \times \mathrm{Aut}\,(C_{p_2^{r_2}}) \times \cdots \times \mathrm{Aut}\,(C_{p_m^{r_m}}) \times \mathrm{Aut}\,(C_{2^s})$.
- For all $\alpha_i \in \mathrm{Aut}\,(C_{p_i^{r_i}})$, $(z_1 \cdots z_{i-1} z_i z_{i+1} \cdots z_n)^{\alpha_i} = z_1 \cdots z_{i-1} z_i^{\alpha_i} z_{i+1} \cdots z_m$, where $z_j \in C_{p_j^{r_j}}$.
- For all $i \neq j$, $[R(C_{p_j^{r_j}}), \mathrm{Aut}\,(C_{p_i^{r_i}})] = [R(C_{2^s}), \mathrm{Aut}\,(C_{p_i^{r_i}})] = 1$.
- $A = R(C_n)\mathrm{Aut}\,(C_n, S) \leq \mathrm{Hol}(C_n) = R(C_n)\mathrm{Aut}\,(C_n)$ and $A_1 = \mathrm{Aut}\,(C_n, S)$, that is because $\Gamma = \mathrm{Cay}(C_n, S)$ is normal.
- Let G be a regular subgroup of A with $G \cong C_n$. To prove that Γ is a CI-digraph, it suffices to show that $G = R(C_n)$.

Proof of Theorem 1.1

- Note that G has a unique Sylow p_i-subgroup G_{p_i} with $|G_{p_i}| = p_i^{r_i}$, and $G \leq A \leq \mathrm{Hol}(C_n) = R(C_n) \rtimes (\mathrm{Aut}\,(C_{p_1^{r_1}}) \times \mathrm{Aut}\,(C_{p_2^{r_2}}) \times \cdots \times \mathrm{Aut}\,(C_{p_m^{r_m}}) \times \mathrm{Aut}\,(C_{2^s}))$.
- **Claim 1.** Assume that $g = R(a)\alpha_1\alpha_2\cdots\alpha_m\alpha_{m+1} \in G$, where $a \in C_n$, $\alpha_{m+1} \in \mathrm{Aut}\,(C_{2^s})$ and $\alpha_i \in \mathrm{Aut}\,(C_{p_i^{r_i}})$ for each $1 \leq i \leq m$, and assume that $R(C_{p_k^{r_k}}) \leq G$ for some $1 \leq k \leq m$. Then $\alpha_k = 1$.
- **Claim 2.** $R(C_{p_k^{r_k}}) \leq G$ for each $1 \leq k \leq m$.
- To prove $G = R(C_n) = R(C_{p_1^{r_1}}) \times R(C_{p_2^{r_2}}) \times \cdots \times R(C_{p_m^{r_m}}) \times R(C_{2^s})$ with $s \leq 2$, we are only left to show $R(C_{2^s}) \leq G$, and it suffices to show that $G_2 \leq R(C_{2^s})$, because $|G_2| = 2^s$.

Proof of Theorem 1.1

- Let $x \in G_2$. Since $G \leq A \leq \mathrm{Hol}(C_n) = R(C_n) \rtimes (\mathrm{Aut}\,(C_{p_1^{r_1}}) \times \cdots \times \mathrm{Aut}\,(C_{p_m^{r_m}}) \times \mathrm{Aut}\,(C_{2^s}))$, we have $x = R(x_1)\gamma_1\gamma_2\cdots\gamma_m\alpha$.
- By Claims 2 and 1, $\gamma_i = 1$ for each $1 \leq i \leq m$, and hence $x = R(x_1)\alpha \in R(C_n)\mathrm{Aut}\,(C_{2^s})$. Since $[R(C_{p_i^{r_i}}), \mathrm{Aut}\,(C_{2^s})] = 1$, $R(C_n)\mathrm{Aut}\,(C_{2^s})$ has a unique Sylow 2-subgroup, that is, $R(C_{2^s})\mathrm{Aut}\,(C_{2^s})$. It follows that $x \in R(C_{2^s})\mathrm{Aut}\,(C_{2^s})$, and hence $G_2 \leq R(C_{2^s})\mathrm{Aut}\,(C_{2^s})$.
- If $\mathrm{Aut}\,(C_{2^s}) = 1$, then $G_2 \leq R(C_{2^s})$, as required.
- If $\mathrm{Aut}\,(C_{2^s}) \neq 1$, and hence $s = 2$ as $s \leq 2$. It follows that $\mathrm{Aut}\,(C_{2^s}) = \mathrm{Aut}\,(C_4) \cong C_2$. In particular, $R(C_{2^s})\mathrm{Aut}\,(C_{2^s})$ is a dihedral group of order 8 and hence has a unique cyclic subgroup of order 4, that is, $R(C_{2^s})$.
- Since $G \cong C_n$, G_2 is a cyclic group of order 4, and since $G_2 \leq R(C_{2^s})\mathrm{Aut}\,(C_{2^s})$, $G_2 = R(C_{2^s})$, as required.

Ideas from S-ring theory

Let $\mathbb{Z}$ be a integer ring and let G be a finite group. Let $\mathbb{Z}G$ be the group ring of G over $\mathbb{Z}$. $\mathbb{Z}G := \{\alpha | \alpha = \sum_{x\in G} a_x \cdot x, a_x \in \mathbb{Z}\}$:

(1) $\sum_{x\in G} a_x \cdot x = \sum_{x\in G} b_x \cdot x \Longleftrightarrow a_x = b_x, \forall x \in G$;

(2) the addition: $\alpha + \beta = (\sum_{x\in G} a_x \cdot x) + (\sum_{x\in G} b_x \cdot x) = \sum_{x\in G} (a_x + b_x) \cdot x$;

(3) the multiplication:
$\alpha\beta = (\sum_{x\in G} a_x \cdot x)(\sum_{y\in G} b_y \cdot y) = \sum_{x,y\in G} a_x b_y \cdot xy = \sum_{z\in G} c_z \cdot z$, where
$c_z = \sum_{xy=z} a_x b_y = \sum_{x} a_x b_{x^{-1}z} = \sum_{y} a_{zy^{-1}} b_y$;

(4) the scalar multiplication: $a\alpha = a(\sum_{x\in G} a_x \cdot x) = \sum_{x\in G} (aa_x) \cdot x$.

Moreover, with these operations, $\forall a \in \mathbb{Z}$, and $\forall \alpha, \beta \in \mathbb{Z}G$, $(a\alpha)\beta = a(\alpha\beta) = \alpha(a\beta)$.

[16] H. Wielandt, Finite permutation groups, Academic Press, New York London (1964).

Ideas from S-ring theory

Let G be a finite group with identity element e and $\mathbb{Z}G$ be the integer group ring. A subring $\mathcal{A} \subseteq \mathbb{Z}G$ is called a Schur ring, an S-ring for short, if it satisfies the following conditions.

(a) $T_0 = \{e\}$,

(b) For each $0 \le i \le r$ the subset $T_i^{(-1)} = \{t^{-1} | t \in T_i\}$ belongs to $\mathcal{T}$,

(c) There exists a partition $\mathcal{T} = \{T_0, T_1, \ldots, T_r\}$ of G such that $\mathcal{A}$ is generated as a vector space by the elements of the following form: $\underline{T}_i = \sum_{t \in T_i} t$.

[24] H. Wielandt, Finite permutation groups, Academic Press, New York London (1964).

Ideas from S-ring theory

- A Cayley digraph $\Gamma = \mathrm{Cay}(G, S)$ of a group G is CI if and only if the smallest S-ring $\mathcal{A}$ over G such that $\underline{S} = \sum_{s \in S} s \in \mathcal{A}$ is CI (see [25] for the definition of a CI-S-ring).
- The automorphism group of any S-ring over G contains the group $R(G)$ and an S-ring is called *normal* if $R(G)$ is normal in its automorphism group.
- Due to [26], we have $\mathrm{Aut}\,(\Gamma) = \mathrm{Aut}\,(\mathcal{A})$. So Γ is normal if and only if $\mathcal{A}$ is normal.
- Normal S-rings over cyclic groups were studied in [27]. In fact, using [26, Corollary 6.5, Theorem 6.6, Lemma 7.1], it is possible to derive that if $8 \nmid n$ then every normal S-ring and hence every normal Cayley digraph over a cyclic group of order n is CI.

[25] M. Hirasaka, M. Muzychuk, An elementary abelian group of rank 4 is a CI-group, J. Combin. Theory Ser. A 94 (2001), 339-362.

[26] G. Chen, I. Ponomarenko, Coherent configurations. Wuhan: Central China Normal University Press, 2019.

[27] S. Evdokimov, I. Ponomarenko, Characterization of cyclotomic schemes and normal Schur rings over a cyclic group, St. Petersburg Math. J. 14 (2003), 189-221.

Dihedral NDCI-group

[28, Theorem 1.1]

Let $n \geq 2$ be a positive integer and let D_{2n} be the dihedral group of order $2n$. Then D_{2n} is a NDCI-group if and only if D_{2n} is a NCI group if and only if either $n = 2, 4$ or n is odd.

[29, Theorem 1.1]

A dihedral group D_{2n} of order $2n$ is a DCI-group if and only if $n = 2$ or n is odd and square-free, and D_{2n} is a CI-group if and only if $n = 2, 9$ or n is odd and square-free.

[28] J.-H Xie, Y.-Q. Feng, J.-X. Zhou, Normal Cayley digraphs of dihedral groups with CI-property, arXiv:2105.12925v2.

[29] J.-H Xie, Cayley digraphs of dihedral groups with CI-property, Working paper. (Y.-Q. Feng, B.Z. Xia)

Thank you for attention.