An introduction to EKR properties of permutation groups

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Outline

1. Erdös-Ko-Rado type problems

2. Erdös-Ko-Rado properties for permutation groups

3. Group-theoretic methods

Erdös-Ko-Rado

Theorem (Erdös-Ko-Rado)

Assume $n \geq 2k$ and let $\mathcal{F}$ be a family of k-subsets of $[n]$ such that any two subsets of $\mathcal{F}$ intersects. Then

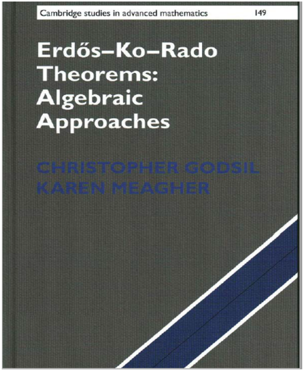
- $|\mathcal{F}| \leq \binom{n-1}{k-1}$
- *$n > 2k$ and $|\mathcal{F}| = \binom{n-1}{k-1} \Rightarrow \mathcal{F}$ consists of all k-subsets containing a given point from $[n]$.*

Erdös-Ko-Rado (abbreviated as EKR) type problem:

- Objects.
- Definition of two objects “intersects“.
- How large can an intersecting family be.

Examples

Objects	Definition of 'intersect'
k-subsets of $[n]$	two subsets intersect if they intersect as sets
Subspaces of V	$W_1, W_2 \subseteq V$ intersect if $W_1 \cap W_2 \neq (0)$
Permutation group G on Δ	$g_1, g_2 \in G$ intersect if $\exists \delta \in \Delta$, $\delta^{g_1} = \delta^{g_2}$



EKR problems for permutation groups

Throughout let G be a transitive permutation group on Δ with stabilizer $H = G_\delta$ for some $\delta \in \Delta$.

Let $S \subseteq G$, TFAE

1. For any $s_1, s_2 \in S$, there exists $\delta \in \Delta$ such that $\delta^{s_1} = \delta^{s_2}$,
2. For any $s_1, s_2 \in S$, there exists $g \in G$ such that $s_1 s_2^{-1} \in H^g$,
3. $SS^{-1} \subseteq \bigcup_{g \in G} H^g$.

S is called an intersecting subset of G if S satisfies above.

Example

1. (Trivial intersecting subsets) H, H^g $H^{g_1} g_2$.
2. $G = \mathrm{L}_2(q), q = p^f$, p is a prime and f is odd. The stabilizer $H \cong C_p$. $S \in \mathrm{Syl}_p(G)$ is an intersecting subset of G.
3. S is an intersecting subset Sg, S^g are intersecting subsets

EKR problems for permutation group.

Let $G \leq \mathrm{Sym}(\Delta)$ be transitive and $H = G_\delta$.

- EKR property: For any intersecting set S we have $|S| \leq |H|$.
- strict EKR property: "EKR property" + Any intersecting set S with $|S| = |H| \Rightarrow S = H^{g_1} g_2$.

Lemma

G has a regular subgroup $\Rightarrow$ G has the EKR property.

Corollary

Frobenius groups have EKR property.

Algebraic graph approach-CS bound

The derangement graph Γ_{dG} of G is defined as follows.

- $V(\Gamma_{dG}) = G$
- $g_1 \sim g_2$ if $g_1 g_2^{-1}$ fixes no point in Δ, (i.e $g_1 g_2^{-1}$ is a **derangement**)

Observation:

- Let $D \subseteq G$ be the sets derangement of G, then $D^G = D$ and $\Gamma_{dG} = \mathrm{Cay}(G, D)$.
- S is an intersecting subset of $G \iff S$ is an independent set in Γ_{dG}.

Lemma

*Let Γ be a vertex-transitive graph, let C be a **clique** of Γ and let S be an **independent** set of Γ, then $|C||S| \leq |V(\Gamma)|$*

Count $\{(g, c) \mid g \in \mathrm{Aut}(\Gamma), c \in C, c^g \in S\}$ in two ways.

Remark $|C| \geq |\Delta| \Rightarrow G$ has the EKR property.

Further examples

Example

The action of $\mathrm{GL}_n(q)$ on $\mathbb{F}_q^n \backslash \{0\}$ has EKR property.

Example

The natural action of S_n on $[n]$ has the EKR property.

Example

The action of $\mathrm{PGL}_n(q)$ on 1-spaces of $\mathbb{F}_q^n$ has EKR property.
Let $\phi_x \in \mathrm{GL}_n(q) : \mathbb{F}_{q^n} \longrightarrow \mathbb{F}_{q^n}, \quad a \mapsto ax$, let $\mathbb{F}_{q^n}^{\times} = \langle \zeta \rangle$, and let $m = \frac{q^n-1}{q-1} = 1 + q + \cdots + q^{n-1}$. Then

$$\{Id, \overline{\phi_\zeta}, \overline{\phi_{\zeta^2}}, \ldots, \overline{\phi_{\zeta^{m-1}}}\}$$

is a clique of the derangement graph of $\mathrm{PGL}_n(q)$ of size $|\Delta|$.

The ratio bound.

Lemma

Let Γ be a k-regular graph with least eigenvalue τ and let $\alpha(\Gamma)$ ***independence number****. Then $\alpha(\Gamma)(1 - \frac{k}{\tau}) \leq |V(\Gamma)|$.*

Example

The Kneser graph $\Gamma = K(n, k)$ is defined as follows

- $V(\Gamma) = k$-subsets of $\{1, 2, \ldots, n\}$
- $s_1 \sim s_2 \iff s_1 \cap s_2 = \emptyset$

Then $\mathcal{F}$ is intersecting $\iff$ $\mathcal{F}$ is an independent set of $K(n, k)$.

The minimal eigenvalue of Γ is $-\binom{n-k-1}{k-1}$, hence

$\alpha(\Gamma) \leq \frac{\binom{n}{k}}{1 - \binom{n-k}{k} / -\binom{n-k-1}{k-1}} = \binom{n-1}{k-1}$

The ratio bound

Observation Let $D = \{g \in G \mid g \text{ fixes no point in } \Delta\} \subseteq G$, the adjacency matrix of Γ_{dG} is the matrix of the multiplication of $\sum_{g \in D} g \in Z(\mathbb{C}G)$ on $\mathbb{C}G$ with respect to the basis $\{g \mid g \in G\}$.

Lemma

The eigenvalues of Γ_{dG} *are* $\eta_\chi = \frac{1}{\chi(id)} \sum_{g \in D} \chi(g)$, $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$.

All 2-transitive groups have EKR property.

- If G is 2-transitive then Either G has a regular subgroup, or G is almost simple.
- Let K be a transitive subgroup of G and K has the EKR property, then G has the EKR property (Lem 3.3 of Tiep 2015)
- We may assume G is a simple 2-transitive group.
- Let π be the permutation character of G and χ_0 be the principal character of G, then $\pi = \chi_0 + \psi$, where ψ is irreducible and $\psi(g) = \text{fix}(g) - 1$.
- The eigenvalues of Γ_{dG} afforded by ψ is $\eta_\psi = \frac{1}{n-1}(-1)|D|$.
- If η_ψ is the minimal eigenvalue of Γ_{dG}, then we have $\alpha(\Gamma_{dG}) \leq \frac{|G|}{1-\frac{|D|}{\eta_\psi}} = \frac{|G|}{n}$, then G has the EKR property.

Line	Group S	Degree	Condition on G	Remarks
1	$\mathrm{Alt}(n)$	n	$\mathrm{Alt}(n) \le G \le \mathrm{Sym}(n)$	$n \ge 5$
2	$\mathrm{PSL}_n(q)$	$\frac{q^n-1}{q-1}$	$\mathrm{PSL}_n(q) \le G \le \mathrm{P\Gamma L}_n(q)$	$n \ge 2$, $(n,q) \ne (2,2),(2,3)$
3	$\mathrm{Sp}_{2n}(2)$	$2^{n-1}(2^n-1)$	$G = S$	$n \ge 3$
4	$\mathrm{Sp}_{2n}(2)$	$2^{n-1}(2^n+1)$	$G = S$	$n \ge 3$
5	$\mathrm{PSU}_3(q)$	q^3+1	$\mathrm{PSU}_3(q) \le G \le \mathrm{P\Gamma U}_3(q)$	$q \ne 2$
6	$\mathrm{Sz}(q)$	q^2+1	$\mathrm{Sz}(q) \le G \le \mathrm{Aut}(\mathrm{Sz}(q))$	$q = 2^{2m+1}$, $m > 0$
7	$\mathrm{Ree}(q)$	q^3+1	$\mathrm{Ree}(q) \le G \le \mathrm{Aut}(\mathrm{Ree}(q))$	$q = 3^{2m+1}$, $m > 0$
8	M_n	n	$M_n \le G \le \mathrm{Aut}(M_n)$	$n \in \{11,12,22,23,24\}$, M_n Mathieu group, $G = S$ or $n = 22$
9	M_{11}	12	$G = S$	
10	$\mathrm{PSL}_2(11)$	11	$G = S$	
11	$\mathrm{Alt}(7)$	15	$G = S$	
12	$\mathrm{PSL}_2(8)$	28	$G = \mathrm{P\Sigma L}_2(8)$	
13	HS	176	$G = S$	HS Higman-Sims group
14	Co_3	276	$G = S$	Co_3 third Conway group

TABLE 1. Finite 2-transitive groups of almost simple type

The 2-transitive action of $\mathrm{Sz}(q)$.

Let $G = \mathrm{Sz}(q)$ where $q = 2^e$, let $r = \sqrt{2q}$, $C_4 \cong \langle \rho \rangle \in G$, $C_{q-1} \cong \langle \xi_0 \rangle \leq G$, $C_{q+r+1} \cong \langle \xi_1 \rangle \leq G$, $C_{q-r+1} \cong \langle \xi_2 \rangle \leq G$

	1	ρ^2	ρ	ρ^{-1}	ξ_0^t	ξ_1^t	ξ_2^t
X	q^2	0	0	0	1	-1	-1
X_i	q^2+1	1	1	1	$\epsilon_0^i(\xi_0^t)$	0	0
Y_j	$(q-r+1)(q-1)$	$r-1$	-1	-1	0	$-\epsilon_1^j(\xi_1^t)$	0
Z_k	$(q+r+1)(q-1)$	$-r-1$	-1	-1	0	0	$-\epsilon_2^k(\xi_2^t)$
W_ℓ	$\frac{r(q-1)}{2}$	$\frac{-r}{2}$	$\frac{r\sqrt{-1}}{2}$	$\frac{r\sqrt{-1}}{2}$	0	1	-1

Table: Character table of $\mathrm{Sz}(q)$

$$\text{where } \epsilon_0^i(\xi_0^t) = \zeta_{q-1}^{it} + \zeta_{q-1}^{-it},\ t \in \{1, 2, \ldots, q-2\}$$
$$\epsilon_1^j(\xi_1^t) = \zeta_{q+r+1}^{jt} + \zeta_{q+r+1}^{jtq} + \zeta_{q+r+1}^{-jt} + \zeta_{q+r+1}^{-jtq},\ t \in \{1, 2, \ldots, q+r\}$$
$$\epsilon_1^k(\xi_2^t) = \zeta_{q-r+1}^{kt} + \zeta_{q-r+1}^{ktq} + \zeta_{q-r+1}^{-kt} + \zeta_{q-r+1}^{-ktq},\ t \in \{1, 2, \ldots, q-r\}$$

Permutation groups without EKR property

Theorem (K.Maegher, A Sarobidy Razafimahatratra 2021)

The action of $\mathrm{AGL}_2(q)$ *on affine lines does not have the EKR property.*

Observation: A block system of affine lines: ℓ_1, ℓ_2 is in the same block $\iff \ell_1 \parallel \ell_2$.

- Let $\mathrm{AGL}_2(q)_B$ be the kernel of $\mathrm{AGL}_2(q)$ on blocks, $\mathrm{AGL}_2(q)_B = \{(\lambda I_d, z) \mid \lambda \in \mathbb{F}_q^\times, z \in \mathbb{F}_q^2\}$
- $\mathrm{AGL}_2(q)_B$ is an intersecting set
- If $(M, z) \in \mathrm{AGL}_2(q)$ fixes two blocks, then (M, z) fixes a line.
- let S be a 2-**intersecting set** of $\mathrm{AGL}_2(q)$ on blocks. (i.e $s_1 s_2^{-1}$ fixes 2 blocks of) Then
$$\bigcup_{s \in S} \mathrm{AGL}_2(q)_B s$$
is an intersecting set of $\mathrm{AGL}_2(q)$.
- There is a 2 intersecting set with size larger than $\frac{3q-5}{2}$.

$\mathrm{L}_2(p^f)$ with stabilizer C_p

Let $G = \mathrm{L}_2(p^f)$ with stabilizer $H \cong C_p$

- f is odd or $p = 2 \Rightarrow \mathbb{F}_{p^f}^{+} \cong Q \in \mathrm{Syl}_p(G)$ is an intersecting subgroup of G.
- If p is odd and f is even, we identify Q with $\mathbb{F}_{p^f}^{+}$, let Q_1 be $\mathbb{F}_{p^{f/2}}^{+}$ under this identification, Q_1 is an intersecting subgroup of G.

Maximal intersecting set: An intersecting set with maximal cardinality.
Intersecting density: $\rho(G) = \frac{|S|}{|G_\delta|}$ with S a maximal intersecting set .

Lemma (Something like Ademir Hujdurović Theorem 7.2, 2022)

- *f is odd or $p = 2 \Rightarrow \rho(G) = p^{f-1}$,*
- *f is even, p is odd and $f > 2 \Rightarrow \rho(G) = p^{f/2-1}$.*

$\mathrm{L}_2(p^f)$ with stabilizer C_p

Lemma

f is odd or $p = 2 \Rightarrow \rho(G) = p^{f-1}$. *f is even, p is odd and* $f > 2 \Rightarrow \rho(G) = p^{f/2-1}$

proof: Let $P_1, P_2, \ldots, P_{p^f+1}$ be $p^f + 1$ Sylow-p subgroup of G, assume

$$P_1 = \left\{ \overline{\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}} \mid x \in \mathbb{F}_{p^f} \right\}, \text{ and } P_2 = \left\{ \overline{\begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix}} \mid y \in \mathbb{F}_{p^f} \right\}$$

- Let S be a maximal intersecting set and $1 \in S$, we assume $P_1 \cap S \neq 1$ and $1 \neq X = \overline{\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}} \in P_1 \cap S$ up to a conjugation of S.
- $Y \in S \Rightarrow Y^p = 1 \Rightarrow$ the preimage of Y in $\mathrm{SL}_2(p^f)$ is of trace ± 2.
- Assume $Y \in S \backslash P_1$ and $Y = \overline{\begin{bmatrix} a & b \\ c & 2-a \end{bmatrix}}$, $p \neq 2$, $YX^{-1} \in S \Rightarrow$ $\mathrm{Tr}(\begin{bmatrix} a & b - ax \\ c & 2 - a - cx \end{bmatrix}) = -2 \Rightarrow c = 4/x \Rightarrow |S \cap P_1 \backslash 1| \leq 1$.

$\mathrm{L}_2(p^f)$ with stabilizer C_p

- $p = 2,\ YX^{-1} \in S \Rightarrow Y \in P_1 \Rightarrow S \subseteq P_1$.
- $p \neq 2, S \nsubseteq P_1 \Rightarrow |(S \cap P_i)\backslash 1| \leq 1$, if $|S| \geq 3$, up to a conjugation we may assume $S \cap P_1 \neq 1$ and $S \cap P_2 \neq 1$.
- We may assume $X = \overline{\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}} \in S \cap P_1$ and $Y = \overline{\begin{bmatrix} 1 & 0 \\ 4/x & 1 \end{bmatrix}} \in S \cap P_2$.
- $Z \in S \backslash (X \cap Y \cap 1)$, we may assume $Z = \overline{\begin{bmatrix} a & b \\ 4/x & 2-a \end{bmatrix}}$, $Z \notin P_1, Z \notin P_2$ and $ZY^{-1} \in S \Rightarrow b = x$.
- $\begin{bmatrix} a & x \\ 4/x & 2-a \end{bmatrix} \in \mathrm{SL}_2(q) \Rightarrow Z$ has at most 2 choices and $|S| \leq 5$,
- $|S| \leq 5$ or $S \subseteq P_1$

$\mathrm{L}_2(p^f)$ with stabilizer C_p

Assume $S \subseteq P_1$

- f is odd, all cyclic subgroup of $G = \mathrm{L}_2(p^f)$ are conjugate and $S = P_1$
- f is even, there are two G classes of cyclic subgroup of order p C_1, C_2 with $C_1 \cap P_1 = \left\{ \overline{\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}} \mid 0 \neq x \in \mathbb{F}_{p^f}^{\square} \right\}$
- If f is even, $\mathbb{F}_{p^{f/2}}^{\times}$ contains all square elements of $\mathbb{F}_{p^f}^{\times}$, therefore $S = \left\{ \overline{\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}} \mid x \in \mathbb{F}_{p^{f/2}} \right\}$ is an intersecting subset of G.
- If f is even, recall Paley graph $P(p^f)$ is defined by $V(P(p^f)) = \mathbb{F}_{p^f}$, $x_1 \sim x_2$ in $P(p^f)$ if $x_1 - x_2 \in \mathbb{F}_{p^f}^{\square}$.
- S is a clique of $P(p^f)$, $|S| \leq \omega(P(p^f)) = p^{f/2}$, where $\omega(P(p^f))$ is the clique number of $P(p^f)$.

Permutation groups without EKR property

Example

Let $G \cong \mathrm{Sz}(q)$ with stabilizer $H \cong C_4$, then the Sylow-2 group of G is an intersecting subset of G.

Example (Ademir Hujdurović et. al 2022)

Let $G \cong \mathrm{L}_2(p^e)$ where $p^e \equiv 1 \pmod 3$ and $H \cong C_3$, then $\rho(G) = 4/3$ if $p \neq 5$ and $\rho(G) = 5$ if $p = 5$.

About the strict EKR property

Groups with the strict EKR property.

- The natural action of S_n, A_n (Cameron 2003, B. Ahmadi K. Maegher 2014)
- $\mathrm{L}_2(q)$ on 1-spaces of $\mathbb{F}_q^2$. (Q. Xiang et. al 2018)
- $G = \mathrm{GL}_2(q)$ or $G = \mathrm{SL}_2(q)$,
  $H = \{\text{upper triangular unipotent matrices}\}$. (M. Bardestani, K. Mallahi-Karai Theorem 5, 2014)

Groups with the EKR property and without the strict EKR property

G is the Heisenberg group $G = \left\{ \eta(x,y,z) := \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} := x, y, z \in \mathbb{F}_p \right\}$,
$H = \{\eta(x,0,0) : x \in \mathbb{F}_p\}, S := \{\eta(x,0,x^2) : x \in \mathbb{F}_p\}$ is an intersecting subset . (example from M. Bardestani, K. Mallahi-Karai, 2014)

About the strict EKR property

$\mathrm{GL}_2(q)$ on $\mathbb{F}_q^2 \backslash (0)$ does not have the strict EKR property.

$$H = \left\{ \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \mid b \in \mathbb{F}_q^\times \right\}$$

is a stabilizer,

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix} \mid b \in \mathbb{F}_q^\times \right\}$$

is an intersecting set.

Intersecting subgroups and the weak EKR property

An intersecting subgroup of G is an intersecting subset that is also a subgroup of G, TFAE

- $K \leq G$ is an intersecting subgroup.
- $K \subseteq \bigcup_{g \in G} H^g$.
- K contains no derangement of G.

Example

- $G = \mathrm{P\Gamma L}_2(2^f)$, where f is odd, $H = (D_{2(2^f-1)}){:}\langle\phi\rangle$, $K = (2^f{:}(2^f-1)){:}\langle\phi\rangle$ is an intersecting subgroup of G.
- $G = \mathrm{P\Gamma L}_2(2^f)$ where f is even, $H = (D_{2(2^f-1)}){:}\langle\phi\rangle$, $K = \mathrm{L}_2(2^{f/2}){:}\langle\phi\rangle$ is an intersecting subgroup of G.

Problem For almost simple primitive permutation group G, characterize intersecting subgroups K such that $|K| \geq |H|$.

Thank you!

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