Non-commuting, non-generating graphs and intersection graphs of groups

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SUSTech Group Theory Seminar
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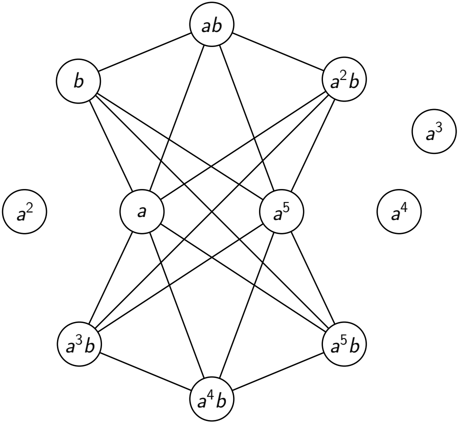

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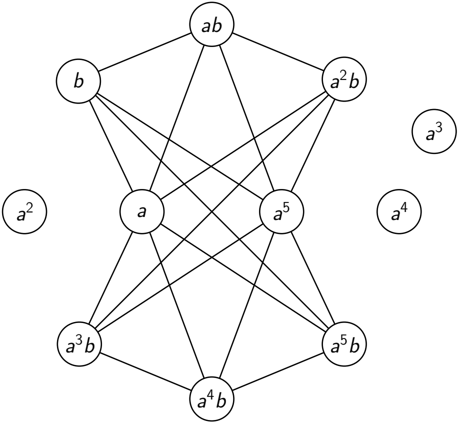

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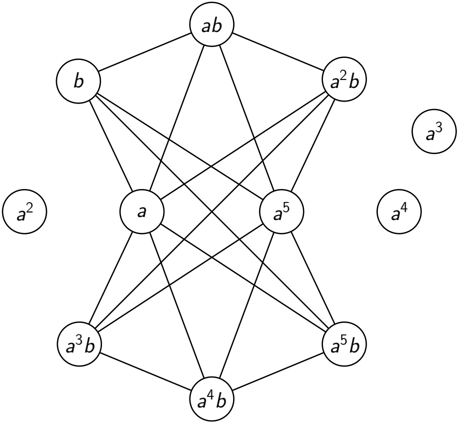

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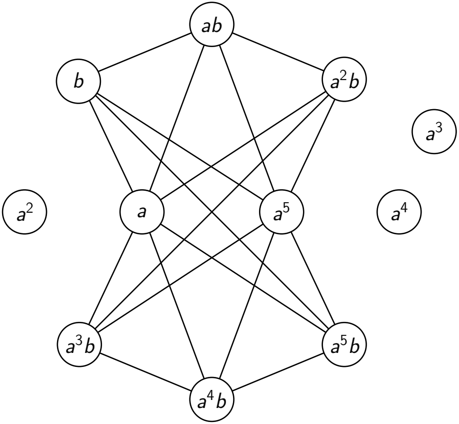

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The non-isolated vertices form a connected component – a subgraph that is maximal among connected subgraphs.

This connected component has diameter 2 – this is the maximal length of a shortest path between two vertices.

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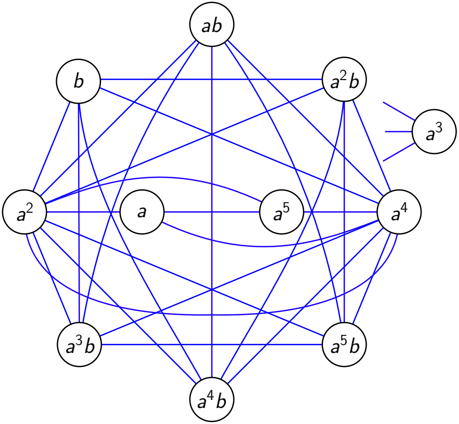

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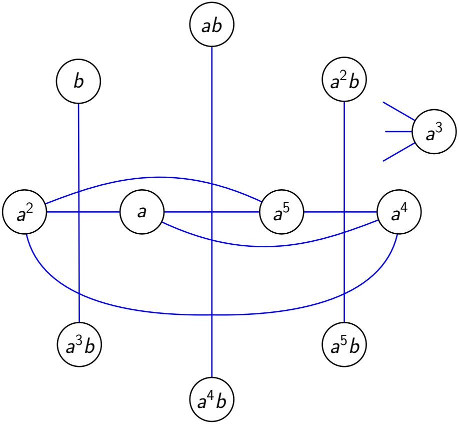

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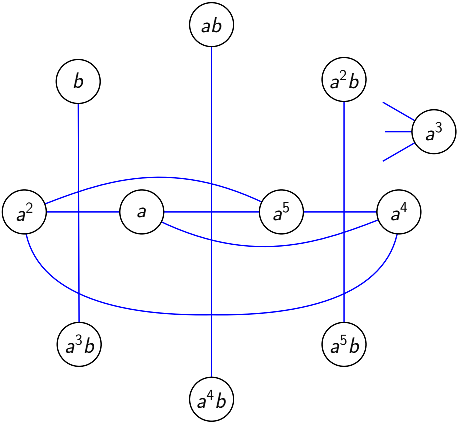

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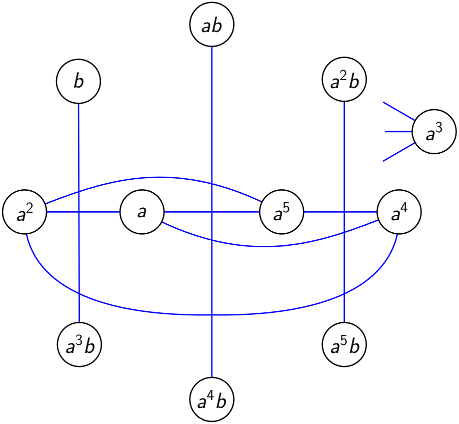

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The generating graph is the difference between the first two graphs.
We will consider the next difference.

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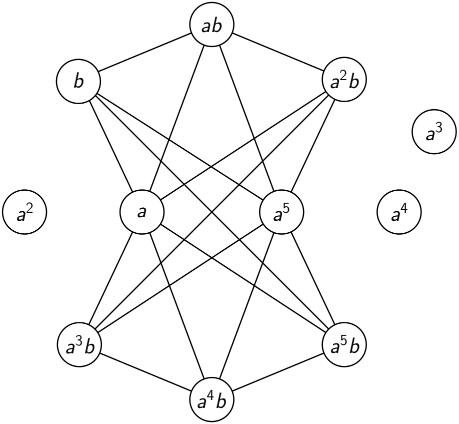

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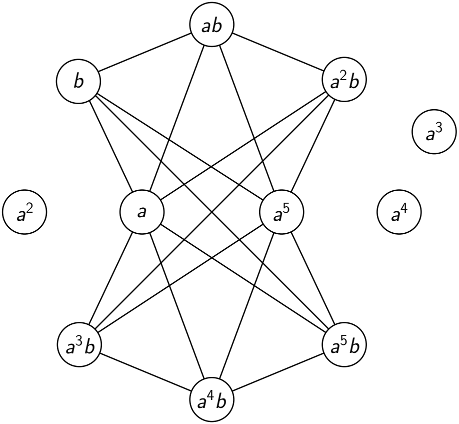

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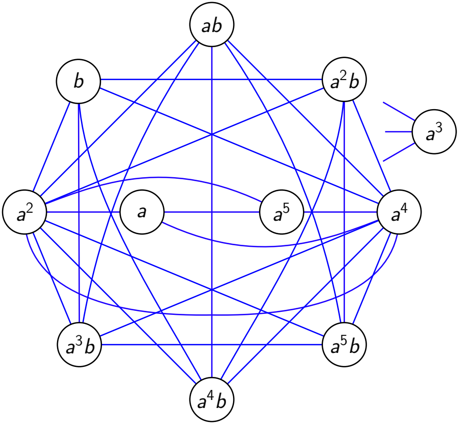

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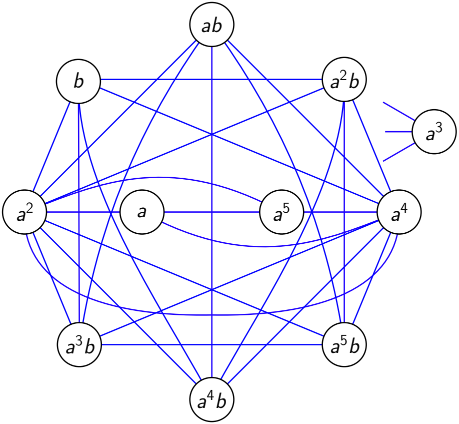

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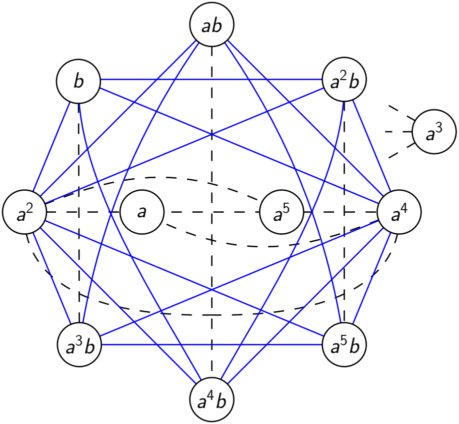

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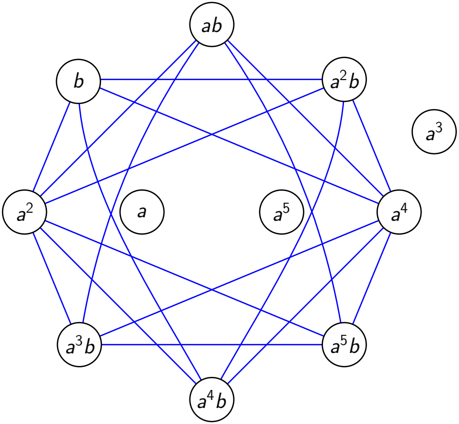

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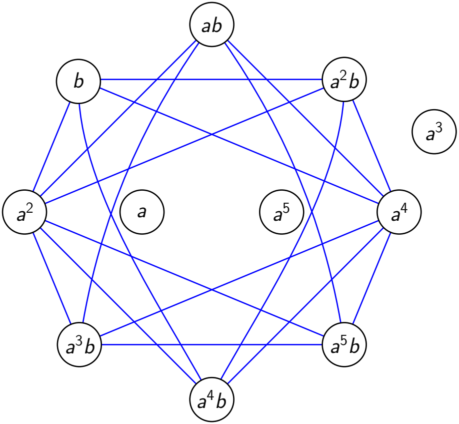

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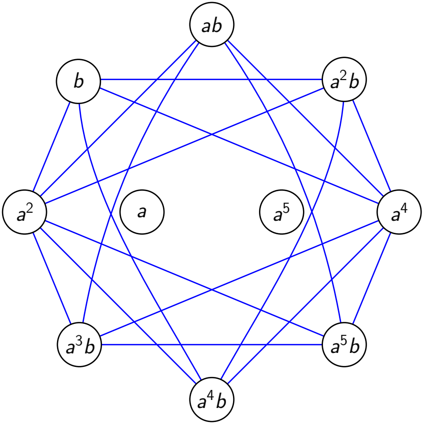

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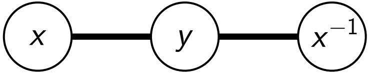

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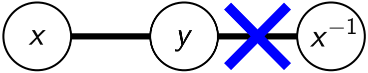

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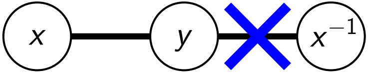

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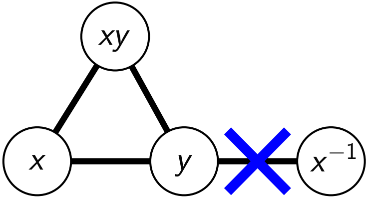

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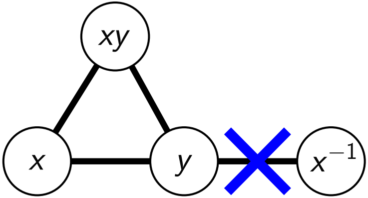

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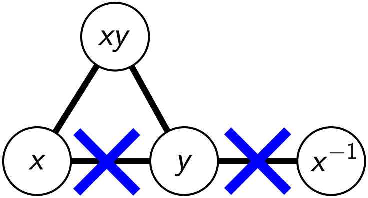

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> Ol'shanskiĭ showed in 1982 that a Tarski monster exists for each prime $p > 10^{75}$.

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If $d \geqslant 3$, then G has no generating pairs. Hence $\Gamma(G)$ is the non-commuting graph of G (with vertices $G \setminus Z(G)$).

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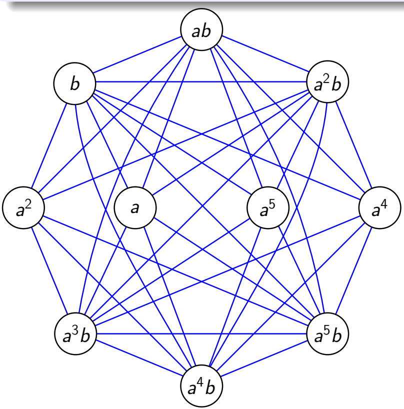

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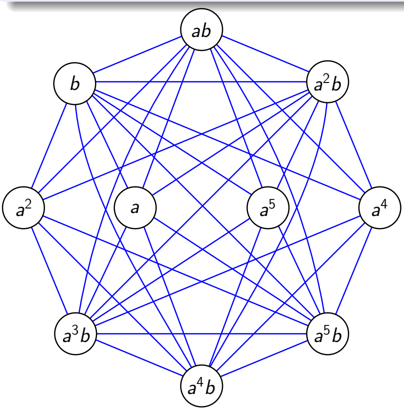

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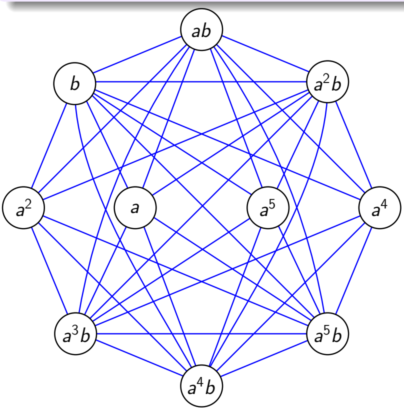

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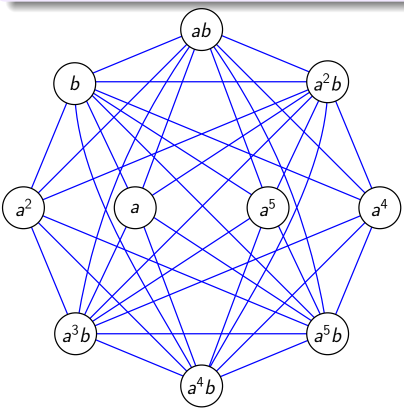

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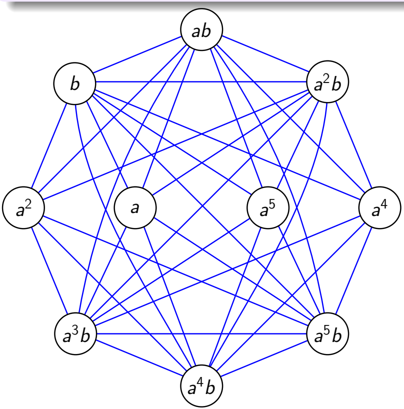

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Similarly to $\Gamma(G)$, the graph cannot have diameter 1. □

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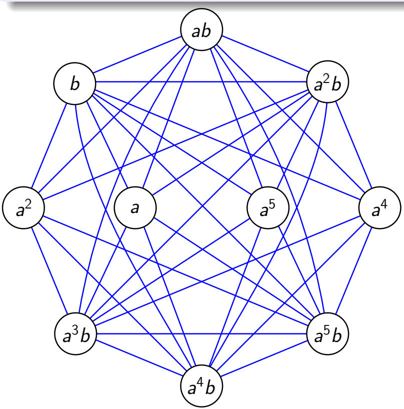

If $x, y \in G \setminus Z(G)$, then $C_G(x) < G$ and $C_G(y) < G$.

The union of two proper subgroups of G is a proper subset of G, so $\exists h_{x,y} \in G \setminus (C_G(x) \cup C_G(y))$.

$(x, h_{x,y}, y)$ is a path in the graph.

Similarly to $\Gamma(G)$, the graph cannot have diameter 1. $\square$

We are therefore only interested in $\Gamma(G)$ when G is 2-generated and non-abelian.

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Thus the above induced subgraph is the non-commuting graph of H, of diameter 2.

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For finite groups, it suffices to prove the conjecture for primitive groups G with all proper quotients cyclic.

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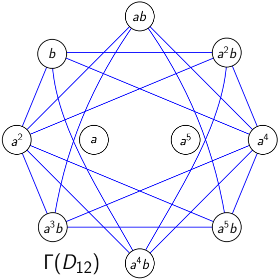

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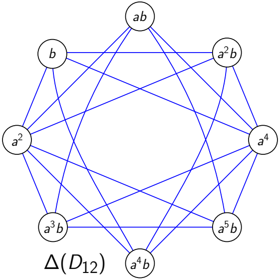

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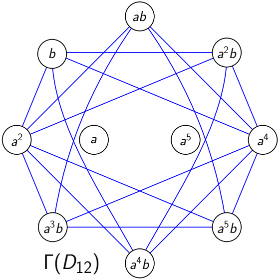

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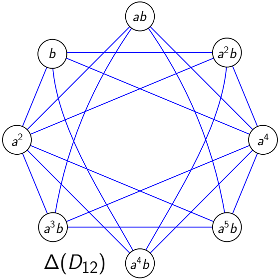

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- $\Gamma(S_4)$ is connected with diameter 3.
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Theorem (Crestani & Lucchini, 2013)

Let k be a positive integer. There exists a non-abelian finite simple group T and a positive integer n such that, excluding isolated vertices, the generating graph of T^n is connected with diameter greater than k.

Soluble groups

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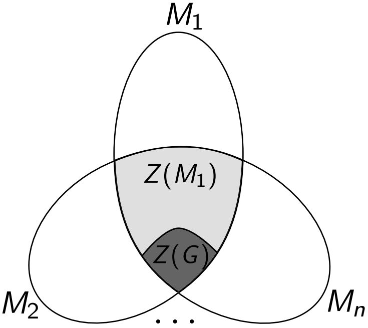

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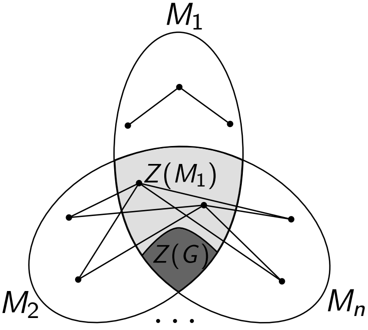

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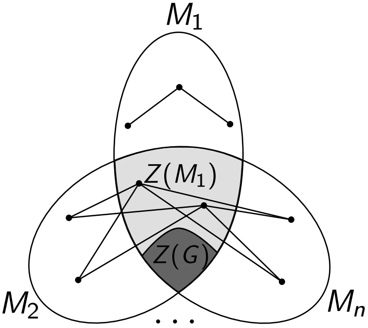

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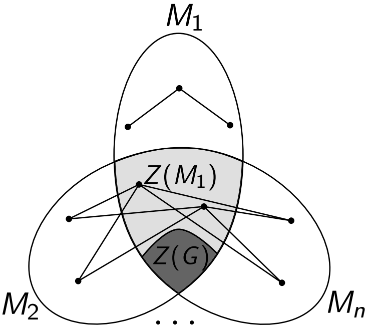

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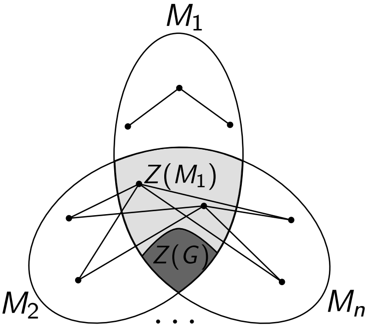

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Finite $[2,2]$-groups

Theorem (Adnan, 1980)

A finite group G has exactly two conjugacy classes of maximal subgroups if and only if there exist distinct primes p and r such that:

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There exists a prime divisor r of $q-1$ such that the subgroup $H \rtimes C_r$ of N is a $[2,2]$-group.

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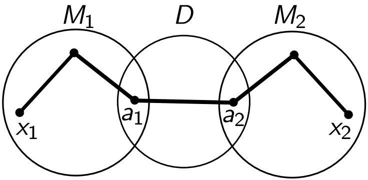

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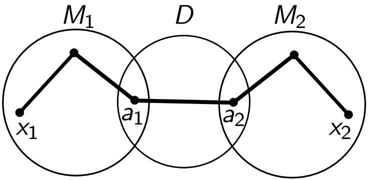

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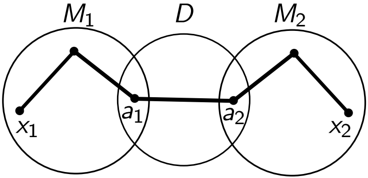

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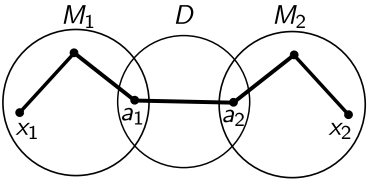

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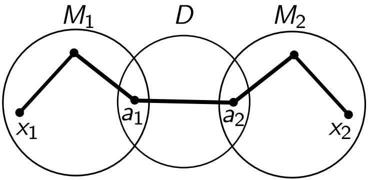

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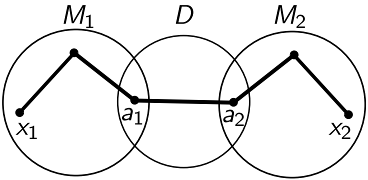

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$\mathrm{M}_{23}, \mathrm{J}_1$	3
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Remaining sporadic groups	$\leqslant 4$
A_n; n even	$\leqslant 3$
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Question: Can these upper bounds be reduced?

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Definition (Csákány & Pollák, 1969)

The intersection graph Σ_G of G has vertices the proper nontrivial subgroups of G, with vertices H and K joined if and only if $H \cap K \neq 1$.

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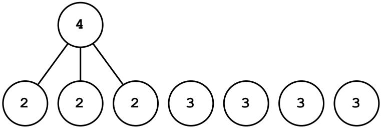

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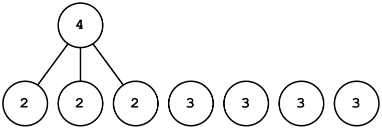

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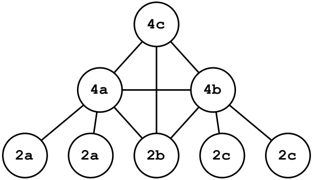

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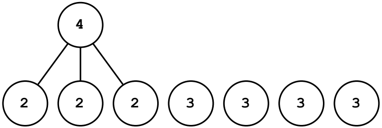

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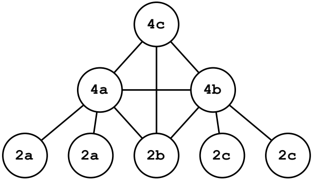

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Let G be a nontrivial, non-simple finite group.

(i) Σ_G is disconnected if and only if $G \cong C_p \times C_q$ for primes p and q; or $Z(G) = 1$ and G is minimal non-abelian.

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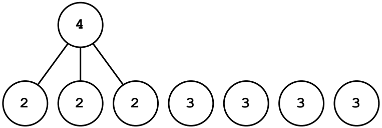

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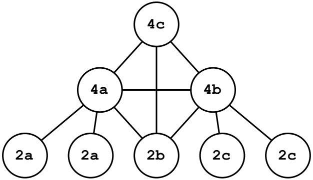

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Open question: Is there a finite non-simple group G with $\mathrm{diam}(\Sigma_G) = 4$? If yes, then $G = S \rtimes C_p$ for a non-abelian simple group S and an odd prime p (Csákány & Pollák, 1969).

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$\mathbb{B}$: $\mathrm{diam}(\Sigma_G) = 5$, by a counting argument involving maximal subgroups.

$\mathbb{M}$? No maximal subgroups of odd order (Holmes & Wilson, 2004, 2008).

$\mathrm{PSL}(n, q)$, n prime: $\mathrm{diam}(\Sigma_G) \leqslant 4$, using arguments from Peter Cameron, involving the group's action on one-dimensional subspaces of $\mathbb{F}_q^n$.

$\mathrm{PSU}(n, q)$, n odd prime: $\mathrm{diam}(\Sigma_G) \leqslant 5$, via similar arguments to the linear case.

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- The soluble graph of $\mathbb{B}$ has diameter 4 or 5 (Burness, Lucchini & Nemmi, 2021+).