Two-geodesic transitive graphs of order p^n with $n \leq 3$

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For a connected, undirected and simple graph Γ with vertex set $V(\Gamma)$, let $u, v \in V(\Gamma)$.

- The **girth** of Γ is defined as the length of a shortest cycle in Γ.
- The **distance** $d_\Gamma(u, v)$ between u and v in Γ is the smallest length of paths between u and v;
- The **diameter** $\text{diam}(\Gamma)$ of Γ is the maximum distance occurring over all pairs of vertices.
- A **geodesic** from a vertex u to v in Γ is one of the shortest paths from u to v, and this geodesic is called an s-**geodesic** if the distance $d_\Gamma(u, v) = s$.
- A sequence $v_0, v_1, \ldots, v_s$ of $s + 1$ vertices of Γ is called an s-**arc** of Γ if v_{i-1}, v_i are adjacent for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s - 1$.
- For a positive integer i, denote by $\Gamma_i(u)$ the set of vertices at distance i with vertex u in Γ. In particular, $\Gamma_1(u)$ is also denoted by $\Gamma(u)$.

Introduction

- Let Γ be vertex transitive graph and let $G \leq \text{Aut}(\Gamma)$. Then
- Γ is said to be (G, s)-**arc transitive** if G is transitive on the set of all s-arcs of Γ.
- Γ is said to be (G, s)-**geodesic transitive** if for each $i \leq s$, G is transitive on the set of all i-geodesics of Γ. If $s = \text{diam}(\Gamma)$, then (G, s)-geodesic transitive graph is called G-**geodesic transitive**.
- Γ is said to be (G, s)-**distance-transitive** if $s \leq \text{diam}(\Gamma)$, and for each $1 \leq i \leq s$, the group G is transitive on $\Gamma_i(u)$. If $s = \text{diam}(\Gamma)$, then (G, s)-distance transitive graph is called G-**distance transitive**.
- In particular, if $G = \text{Aut}(\Gamma)$, then (G, s)-arc transitive, (G, s)-geodesic transitive or (G, s)-distance-transitive graph is simply called an s-**arc transitive**, s-**geodesic transitive** or s-**distance-transitive**, respectively.

Let Γ be a 2-arc transitive graph. Then either $\Gamma \cong \mathrm{K}_n$, or Γ has girth at least 4.

Let Γ be a 2-geodesic transitive graph.

- If Γ has girth at least 4, then Γ is 2-arc transitive.
- If Γ has girth 3, then Γ is not 2-arc transitive except K_n.

Let Γ be a 2-distance transitive graph.

- If Γ has girth at least 5, then Γ is 2-arc transitive.
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By definitions, it is clear that we have a hierarchy of conditions:

s-arc transitive $\Rightarrow$ s-geodesic transitive $\Rightarrow$
s-distance transitive $\Rightarrow$ arc transitive.

However, both the inverse are not true.

For a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the **Cayley graph** $\mathrm{Cay}(G, S)$ on G with respect to S is defined to be the graph with vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$.

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Example: arc-transitive but not 2-distance transitive graphs

Let p be an odd prime and let r be a positive integer such that r is a divisor of $p-1$. Let H and H' denote two disjoint copies of the additive group $\mathbb{Z}_p$ and denote the corresponding elements of H and H' by i and i'. Denote the unique subgroup of order r of the multiplicative group of $\mathbb{Z}_p$ by $H(p,r)$. We define $G(2p,r)$ to be the graph with vertex set $H \cup H'$ and edge set $\{\{i,j'\} \mid i,j \in \mathbb{Z}_p, j-i \in H(p,r)\}$.

- Observe that $G(2p,1) \cong p\mathsf{K}_2$ and $G(2p,p-1) \cong \mathsf{K}_{p,p}$.
- If $1 < r < p-1$ and $(p,r) \neq (7,3)$, $(11,5)$, then $G(2p,r)$ is a connected arc transitive dihedrant and $\mathrm{Aut}(G(2p,r)) \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_r) \rtimes \mathbb{Z}_2$, see [40, Lemma 2.1 and Table 1].
- $G(2p,r)$ is a connected 2-distance transitive graph if and only if Γ is 2-arc transitive and one of the following holds:
  - (i) $r=2$ and $\Gamma \cong \mathbf{C}_{2p}$;
  - (ii) $r=p-1$ and $\Gamma \cong \mathsf{K}_{p,p}$;
  - (iii) $(p,r)=(7,3)$ or $(11,5)$, and $\Gamma \cong B(\mathsf{PG}(2,2))$ or $B(H_{11})$.

[40] Y. Cheng and J. Oxley, On weakly symmetric graphs of order twice a prime, J. Combin. Theory Ser. B 42 (1987), 196-211.

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Example: 2-distance transitive but not 2-geodesic transitive graphs

Let $q = p^e$ be a prime power such that $q \equiv 1 (\mathrm{mod}\ 4)$. Let F_q be the finite field of order q. The **Paley graph** $\mathrm{P}(q)$ is the graph with vertex set F_q, and two vertices u, v are adjacent if and only if $u - v$ is a nonzero square in F_q. Furthermore, let $N = F_q^+ \cong \mathbb{Z}_p^e$ be the additive group of F_q, and let ω be a primitive element of F_q. Then $\mathrm{P}(q) \cong \mathrm{Cay}(N, S)$ with $S = \langle \omega^2 \rangle$. Moreover, by [3, Theorem 7.1], $\mathrm{Aut}(\mathrm{P}(q)) \cong (F_q^+ \rtimes \langle \omega^2 \rangle).\langle \tau \rangle \cong (\mathbb{Z}_p^e \rtimes \mathbb{Z}_{(q-1)/2}).\mathbb{Z}_e$, where $\tau : x \mapsto x^p$ for any $x \in F_q$.

[2, Theorem 1.2]

Let $\Gamma = \mathrm{P}(q)$ be defined as above. Then $\mathrm{diam}(\Gamma) = 2$ and Γ is 2-distance transitive, and the following statements are true:

(1) Γ is 2-arc transitive if and only if $p = 5$ and $\mathrm{P}(5) \cong \mathbf{C}_5$.

(2) Γ is 2-geodesic transitive but not 2-arc transitive if and only if $q = 9$.

[2] W. Jin, A. Devillers, C. H. Li, C. E. Praeger, On geodesic transitive graphs, Discrete Math. 338 (2015), 168-173.

[3] W. Peisert, All self-complementary symmetric graphs, J. Algebra 240 (2001), 209-229.

Example: 2-distance transitive but not 2-geodesic transitive graphs

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Example: 2-distance transitive but not 2-geodesic transitive graphs

Let p be a prime such that $p \equiv 3(\text{mod } 4)$. Let F_q be the finite field of order q, where $q = p^e \equiv 1(\text{mod } 4)$ and e is even. Suppose that $N = F_q^+ \cong \mathbb{Z}_p^e$ be the additive group of F_q, and λ be a primitive element of F_q. Let

$$S = \{\lambda^i \mid i \equiv 0, 1(\text{mod } 4)\} = \langle\lambda^4\rangle \cup \langle\lambda^4\rangle\lambda.$$

Then the Cayley graph Cay(N, S) is called a **Peisert graph**, denoted by $Pei(q)$. Further, $Pei(q)$ is a connected undirected graph of valency $(q-1)/2$, girth 3 and diameter 2. Moreover, by [3, Theorem 7.1], Aut$(Pei(q)) \cong (F_q^+ \rtimes \langle\lambda^4\rangle).\langle\tau\rangle \cong (\mathbb{Z}_p^e \rtimes \mathbb{Z}_{(q-1)/4}).\mathbb{Z}_e$, where τ is the Frobenius automorphism of F_q, and $q \neq 3^2, 7^2, 23^2$. In particular, P$(9) \cong Per(9)$.

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Let $\Gamma = Pei(q)$ be defined as above. Then Γ is 2-distance transitive but not 2-arc transitive, and Γ is 2-geodesic transitive if and only if $q = 9$.

[3] W. Peisert, All self-complementary symmetric graphs, J. Algebra 240 (2001), 209-229.

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Example: 2-geodesic transitive but not 2-arc transitive graphs

(1) $\mathsf{K}_{m[b]}$, the complete multipartite graph consisting $m \geq 3$ parts of size $b \geq 2$.

(2) $H(d,n)$, the Hamming graph.

[1, Section 9.2] and [2, Proposition 2.2]

Let d, n be two positive integers and $d, n \geq 2$. The Hamming graph $H(d,n)$ is defined as the vertex set $\mathbb{Z}_n^d$, seen as a module over the ring $\mathbb{Z}_n = [0, n-1]$, and two vertices are adjacent if and only if they have exactly one different coordinate. In particular, $H(d,2)$ is also known as d-cube. $H(d,n)$ has diameter d, valency $d(n-1)$, is distance transitive, geodesic transitive and $\mathrm{Aut}(H(d,n)) \cong \mathsf{S}_n \wr \mathsf{S}_d$.

[1] A. E. Brouwer, A. M. Cohen, A. Neumaier, Distance-regular graphs, Springer Verlag, Berlin, Heidelberg, New York, 1989.

[2] W. Jin, A. Devillers, C. H. Li, C. E. Praeger, On geodesic transitive graphs, Discrete Math. 338 (2015), 168-173.

Question 1

To construct s-distance transitive but not s-geodesic transitive graphs for $s \geq 2$.

Question 2

To construct s-distance transitive (or s-geodesic transitive) but not s-arc transitive graphs for $s \geq 2$.

- Weiss proved that there are no finite 8-arc-transitive graphs with valency at least three.
- However, there is no upper bound on s for s-geodesic transitivity or s-distance transitivity, for example, $H(d,n)$.

The structure of $[\Gamma(u)]$ of a 2-geodesic transitive graph are determined in [4], and the authors proved that if Γ is a 2-geodesic transitive graph of valency at least 2, then for a vertex u, either

(1) $[\Gamma(u)] \cong m\mathsf{K}_r$ for some integers $m \geq 2$ and $r \geq 1$; or

(2) $[\Gamma(u)]$ is a connected vertex transitive graph of diameter 2.

- A reduction theorem of the locally connected or disconnected 2-geodesic transitive graph were given in [5,6], respectively.

[4] A. Devillers, W. Jin, C. H. Li, C. E. Praeger, Local 2-geodesic transitivity and clique graphs, J. Combin. Theory Ser. A 120 (2013), 500-508.

[5] W. Jin, Two-geodesic-transitive graphs which are locally connected, Discrete Math. 340 (2017), 637-643.

[6] W. Jin, Finite s-geodesic transitive graphs which are locally disconnected, Bull. Malays. Math. Sci. Soc., 42 (2019), 909-919.

Some known results

- The 2-geodesic transitive graphs Γ with $[\Gamma(u)]$ satisfies the following were classified in [5,7,8,9], respectively.

  (1) $[\Gamma(u)]$ has valency 2, 3 or 4;
  (2) $[\Gamma(u)] \cong \mathbf{C}_n, \overline{\mathbf{C}_n}$ or $\overline{m\mathbf{C}_n}$;
  (3) $[\Gamma(u)]$ is an arc transitive circulant;
  (4) $[\Gamma(u)]$ is self-complementary.

[5] W. Jin, Two-geodesic-transitive graphs which are locally connected, Discrete Math. 340 (2017), 637-643.

[7] A. Devillers, W. Jin, C. H. Li, C. E. Praeger, Line graphs and geodesic transitivity, Ars Math. Contemp. 6(2013), 13-20.

[8] W. Jin, L. Tan, Two-geodesic-transitive graphs which are neighbor cubic or neighbor tetravalent, Filomat 32 (2018), 2483-2488.

[9] W. Jin, L. Tan, Two-geodesic-transitive graphs which are locally self-complementary, Discrete Math. 345 (2022), 112900.

Some known results

- The distance transitive graphs of valency $3 \leq k \leq 13$ are listed in [1, Section 7.5].
- The 2-distance transitive but not 2-arc transitive graphs of valency $k \leq 7$ are classified in [10,11,12].
- The 2-geodesic transitive but not 2-arc transitive graphs of valency 6, $p, 2p$ or $3p$, with p a prime, are classified in [13,14,15,16].
- The 3-geodesic transitive but not 3-arc transitive graphs of valency 3,4 or 5 are classified in [17,18].

[1] A. E. Brouwer, A. M. Cohen, A. Neumaier, Distance-regular graphs, Springer Verlag, Berlin, Heidelberg, New York, 1989.

[10] B. P. Corr, W. Jin, C. Schneider, Finite 2-distance transitive graphs, J. Graph Theory, 86 (2017), 78-91.

[11] W. Jin, T. Li, Finite two-distance-transitive graphs of valency 6, Ars Math. Contemp. 11,(2016) 49-58.

[12] W. Jin, L. Tan, Two-distance-transitve graphs of valency 7, Ars Combin. 126 (2016), 211-220.

[13] W. Jin, W. J. Liu, S. J. Xu, Two-geodesic transitive graphs of six, Discrete Math. 340 (2017) 192-200.

[14] A. Devillers, W. Jin, C.H. Li, C.E. Praeger, Finite 2-geodesic transitive graphs of prime valency, J. Graph Theory 80 (1) (2015) 18-27.

[15] W. Jin, Finite 2-geodesic-transitive graphs of valency twice a prime, European J. Combin. 49 (2015) 117-125.

[16] W. Jin, Finite 2-geodesic transitive graphs of valency $3p$, Ars Combin. 120 (2015) 417-425.

[17] W. Jin, Finite 3-geodesic transitive but not 3-arc transitive graphs, Bull. Aust. Math. Soc. 91 (2015) 183-190.

[18] W. Jin, The pentavalent three-geodesic-transitive graphs, Discrete Math. 341 (2018) 1344-1349.

- The 2-distance transitive, 2-geodesic transitive, 2-arc transitive circulants are classified in [19,20,21], respectively.

Let Γ be a connected 2-distance transitive circulant. Then the following statements holds.

(1) If Γ is 2-arc transitive, then Γ is one of the following graphs: K_n with $n \geq 1$, $\mathbf{C}_n$ with $n \geq 4$, $\mathsf{K}_{\frac{n}{2},\frac{n}{2}}$ with $n \geq 6$, $\mathsf{K}_{\frac{n}{2},\frac{n}{2}} - \frac{n}{2}\mathsf{K}_2$ with $\frac{n}{2} \geq 5$ odd.

(2) If Γ is 2-geodesic transitive but not 2-arc transitive, then $\Gamma \cong \mathsf{K}_{m[b]}$ for some $m \geq 3, b \geq 2$.

(3) If Γ is not 2-geodesic transitive, then Γ is the Paley graph $\mathsf{P}(p)$, where p is a prime and $p \equiv 1 (\mathrm{mod}\ 4)$.

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[19] J. Y. Chen, W. Jin and C. H. Li, On 2-distance-transitive circulants, J. Algebraic Combin. 49 (2019), 179-191.

[20] W. Jin, W. J. Liu and C. Q. Wang, Finite 2-geodesic transitive abelian Cayley graphs, Graphs Combin. 32 (2016), 713-720.

[21] B. Alspach, M. D. E. Conder, M. Y. Xu, A classification of 2-arc-transitive circulants, J. Algebraic Combin. 5 (1996), 83-86.

[19, Question 1.2]

Is there a normal Cayley graph which is 2-distance-transitive, but neither distance-transitive nor 2-arc-transitive?

[22, Theorem 1.2]

For an odd prime p, let $G = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1\rangle$ and $S = \{a^i, b^i \mid 1 \leq i \leq p-1\}$. Then Cay$(G, S)$ is a 2-distance-transitive normal Cayley graph that is neither distance-transitive nor 2-arc-transitive. In particular, Aut(Cay$(G, S)) \cong R(G) \rtimes ((\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \rtimes \mathbb{Z}_2)$.

[19] J. Y. Chen, W. Jin and C. H. Li, On 2-distance-transitive circulants, J. Algebraic Combin. 49 (2019), 179-191.

[22] J.J. Huang, Y.Q. Feng, J.X. Zhou, Two-distance transitive normal Cayley graphs, Ars Math. Contemp. 22 (2022) p.#2.02.

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[22] J.J. Huang, Y.Q. Feng, J.X. Zhou, Two-distance transitive normal Cayley graphs, Ars Math. Contemp. 22 (2022) p.#2.02.

Question 3

To construct s-distance transitive but not distance transitive and s-arc transitive graphs for $s \geq 2$.

Question 4

Classify 2-distance transitive graphs of order p^n.

[23, Theorem 1.2] and [24, Theorem 1.3]

For a prime p and a positive integer n, let Γ be a connected 2-geodesic but not 2-arc transitive graph of order p^n, and let Aut(Γ) be quasiprimitive on $V(\Gamma)$. Then Aut(Γ) is primitive on $V(\Gamma)$ and one of the following holds:

(1) Γ is the Schläfli graph or its complement;

(2) Γ is the Hamming graph $H(s, p^t)$ with $p^t \geq 5$ and $st = n$, or $\overline{H(2, p^t)}$ with $p^t \geq 5$;

(3) Γ is a normal Cayley graph on $\mathbb{Z}_p^n$.

[23, Problem 1.4]

Let Γ be a 2-geodesic transitive graph of prime power order which is not 2-arc transitive. Classify such graphs where Aut(Γ) acts quasiprimitively on $V(\Gamma)$ of affine type.

[23] W. Jin, Two-geodesic transitive graphs of prime power order, Bull. Iranian Math. Soc. 43 (2017) 1645-1655.

[24] W. Jin, Vertex quasiprimitive two-geodesic transitive graphs, https://arxiv.org/abs/2106.12357.

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[24] W. Jin, Vertex quasiprimitive two-geodesic transitive graphs, `https://arxiv.org/abs/2106.12357`.

[25, Theorem 1.2]

Let p be a prime and let Γ be a connected 2-geodesic transitive but not 2-arc transitive normal Cayley graph on $\mathbb{Z}_p^n$ with $n \leq 3$. Then $\mathrm{Aut}(\Gamma)$ is primitive on $V(\Gamma)$ if and only if $\Gamma \cong H(2,3)$ or $H(3,3)$.

[25, Theorem 1.4]

Let p be a prime and let Γ be a connected arc-transitive graph of order p^2. Then Γ is 2-geodesic transitive if and only if one of the following holds:

(1) If Γ is 2-arc transitive, then Γ is isomorphic to $\mathbf{C}_{p^2}$ or K_{p^2};

(2) If Γ is not 2-arc transitive, then Γ is isomorphic to $\mathrm{K}_{p[p]}$, the Hamming graph $H(2,p)$ or its complement $\overline{H(2,p)}$, where $p \geq 3$.

[25] J. J. Huang, Y. Q. Feng, J. X. Zhou, F. G. Yin, Two-geodesic transitive graphs of order p^n with $n \leq 3$, https://arxiv.org/abs/2207.10919.

[25, Theorem 1.5]

Let p be a prime and let Γ be a connected arc transitive graph of order p^3. Then Γ is 2-geodesic transitive if and only if one of the following holds:

(1) If Γ is 2-arc transitive, then Γ is isomorphic to $H(3,2)$, $\mathsf{K}_{4,4}$, $\mathbf{C}_{p^3}$ or K_{p^3};

(2) If Γ is not 2-arc transitive, then Γ is isomorphic to one of the following graphs:

  (i) the Schläfli graph or its complement;
  (ii) $\mathsf{K}_{p^2[p]}$ with $p \geq 2$, or $\mathsf{K}_{p[p^2]}$ with $p \geq 3$;
  (iii) the Hamming graph $H(3,p)$ with $p \geq 3$;
  (iv) the normal Cayley graph $\mathsf{Cay}(G, S_i)$ with $i = 1, 2$, where $G = \langle a, b, c \mid a^p = b^p = c^p = 1, [a,b] = c, [a,c] = [b,c] = 1\rangle$ with $p \geq 3$, $S_1 = \{a^i, b^i \mid i \in \mathbb{Z}_p^*\}$ and $S_2 = \{a^i, b^i, (b^j a b^j)^i \mid i, j \in \mathbb{Z}_p^*\}$.

[25] J. J. Huang, Y. Q. Feng, J. X. Zhou, F. G. Yin, Two-geodesic transitive graphs of order p^n with $n \leq 3$, https://arxiv.org/abs/2207.10919.

- Marušič [41, Theorem 3.4] proved that a vertex transitive graph of order p^n, with $n \leq 3$, is a Cayley graph, and 2-arc transitive Cayley graphs of order p^2 were classified by Marušič [42,Corollary 2.3]. This, together with [43, Corollary 1.3 and 3.5], enables us to obtain the following result.

For an odd prime p, a connected 2-arc transitive graph of order p^2 is $\mathbf{C}_{p^2}$ or K_{p^2}, and a connected 2-arc transitive graph of order p^3 is $\mathbf{C}_{p^3}$, K_{p^3}, or a cover of a complete graph.

- Appealing to the main result of [44] we obtain the classification of 2-arc transitive graph with order prime-cube.

Let p be a prime and let Γ be a connected 2-arc transitive graph of order p^3. Then Γ is isomorphic to $H(3, 2)$, $\mathsf{K}_{4,4}$, $\mathbf{C}_{p^3}$ or K_{p^3}.

[41] D. Marušič, Vertex transitive graphs and digraphs of order p^k, Ann. Discrete Math. 27 (1985) 115-128.

[42] D. Marušič, P. Potočnic, Classifying 2-arc-transitive graphs of oder a product of two primes, Discrete Math. 244 (2002) 331-338.

[43] C.H. Li, Finite s-arc transitive graphs of prime-power order, Bull. London Math. Soc. 33 (2001) 129-137.

[44] S.F. Du, D. Marušič, A.O. Waller, On 2-arc-transitive covers of complete graphs, J. Combin. Theory Ser. B 74 (1998) 276-290.

Primitive Case

A Cayley graph Cay(G, S) is called a **normal** $(X, 2)$-**geodesic transitive** if it is $(X, 2)$-geodesic transitive for a group X such that $R(G) \leq X \leq R(G) \rtimes \mathrm{Aut}(G, S)$.

[26, Theorem 1.2]

Let $\Gamma = \mathrm{Cay}(G, S)$ be a connected normal $(X, 2)$-geodesic transitive Cayley graph. Then one of the following holds:

(1) $\Gamma \cong \mathbf{C}_r$ and $G \cong \mathbb{Z}_r$ for some $r \geq 4$;

(2) $\Gamma \cong \mathrm{K}_{4[2]}$ and $G \cong \mathrm{Q}_8$, the quaternion group, with $S = G \setminus Z(G)$;

(3) There is a prime q and an integer m such that for all $a \in S$, a has order q with $\langle a\rangle^* \subseteq S$ and $\langle a\rangle^* \neq S$, and b has order m for each $b \in S^2 \setminus (S \cup \{1\})$.

[26] A. Devillers, W. Jin, C.H. Li, C.E. Praeger, On normal 2-geodesic transitive Cayley graphs, J. Algebr. Comb. 39 (2014) 903-918.

Lemma 5.2

Let $\Gamma = \mathrm{Cay}(\mathbb{Z}_p^n, S)$ be a connected 2-geodesic but not 2-arc transitive normal Cayley graph on $\mathbb{Z}_p^n$, where p is a prime and $n \geq 1$. Then

(1) $\mathrm{Aut}(\mathbb{Z}_p^n, S)$ contains the center of $\mathrm{Aut}(\mathbb{Z}_p^n)$, and for every positive integer m, if $x \in \Gamma_m(1)$ then $\langle x \rangle^* \subseteq \Gamma_m(1)$;

(2) Let K_1 and K_2 be subgroups of $\mathbb{Z}_p^n$. Assume that the induced subgroups $[K_1]$ and $[K_2]$ of Γ are complete graphs and every vertex in K_1 is adjacent to every vertex in K_2. Then the induced subgraph $[K_1K_2]$ of Γ is a complete graph;

(3) $\mathbb{Z}_p^n$ has a subgroup H such that $H^* \subseteq S$ and $[H]$ is a maximal clique of Γ.

- A complete list of all connected symmetric graphs with orders from 2 to 30 was given in [31]. By Magma, we can obtain all 2-geodesic transitive graphs of order 4,8,9,25 or 27.

[31] M.D.E. Conder, A complete list of all connected symmetric graphs of order 2 to 30, https://www.math.auckland.ac.nz/~conder/symmetricgraphs-orderupto30.txt.

Lemma 5.3

Let $p \geq 5$ be a prime and let $\Gamma = \mathrm{Cay}(\mathbb{Z}_p^2, S)$ be a connected 2-geodesic but not 2-arc transitive normal Cayley graph. Then $\mathrm{Aut}(\Gamma)$ cannot be primitive on $V(\Gamma)$.

Outline of a proof of Lemma 5.3:

- Let $\overline{S} = \{\langle s\rangle^{\#} \mid s \in S\}$ and let $\ell = |\overline{S}|$.
- Let $\overline{A_1}$ be the permutation group of A_1 acting on $\overline{S}$.
- Then $\overline{A_1} \leq \mathrm{PGL}(2,p)$ and $\overline{A_1}$ is transitive on $\overline{S}$.
- Moreover, $3 \leq \ell \leq p-2$, $3(p-2) \leq \ell(p+1-\ell)$ and

$$\ell(p+1-\ell) \bigm| |\overline{A_1}_{p'}|.$$

[27, P. 392-418] and [28, Theorem 2]

Let $p \geq 5$ be a prime.

(1) If $M \lessdot \mathrm{PGL}(2,p)$, then M is one of the following groups: $\mathrm{PSL}(2,p)$, $\mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$, $\mathrm{D}_{2(p-1)}$ with $p \neq 5$, $\mathrm{D}_{2(p+1)}$, or S_4 with $p \equiv \pm 3 (\mathrm{mod}\ 8)$;

(2) If $M \lessdot \mathrm{PSL}(2,p)$, then M is one of the following groups: $\mathbb{Z}_p \rtimes \mathbb{Z}_{(p-1)/2}$, D_{p-1} with $p \geq 13$, D_{p+1} with $p \neq 7$, A_4 with $p \equiv \pm 3 (\mathrm{mod}\ 8)$ and $p \equiv \pm 3 (\mathrm{mod}\ 10)$, S_4 with $p \equiv \pm 1 (\mathrm{mod}\ 8)$, or A_5 with $p \equiv \pm 1 (\mathrm{mod}\ 10)$.

[27] M. Suzuki, Group Theory I, Springer, New York, 1982.

[28] P.J. Cameron, G.R. Omidi, B. Tayfeh-Rezaie, 3-Designs from $\mathrm{PGL}(2,q)$, Electron. J. Combin. 13 (2006) #R50.

Lemma 5.4

Let $p \geq 5$ be a prime and let $\Gamma = \mathrm{Cay}(\mathbb{Z}_p^3, S)$ be a connected 2-geodesic but not 2-arc transitive normal Cayley graph. Then $\mathrm{Aut}(\Gamma)$ cannot be primitive on $V(\Gamma)$.

Outline of a proof of Lemma 5.4:

- Let $\overline{S} = \{\langle s\rangle^{\#} \mid s \in S\}$ and let $\ell = |\overline{S}|$.
- Let $\overline{A_1}$ be the permutation group of A_1 acting on $\overline{S}$.
- Then $\overline{A_1} \leq \mathrm{PGL}(3,p)$ and $\overline{A_1}$ is transitive on $\overline{S}$.

Outline of a proof the main results

Case 1: $H \cong \mathbb{Z}_p^2$.

- In the case, $\overline{A_1} \leq \mathrm{PGL}(3,p)$, $7 \leq p+2 \leq \ell \leq p^2+p-3$, and

$$f(\ell,p) := \ell(p^2+p+1-\ell) \text{ is a divisor of } |\overline{A_1}|.$$

[29, Tables 8.3 and 8.4] and [30, Chapter 4]

Let $p \geq 5$ be a prime and let $d=(3,p-1)$.

(1) If $M \lessdot \mathrm{PGL}(3,p)$, then M is one of the following groups: $\mathrm{PSL}(3,p)$, $\mathbb{Z}_p^2:\mathrm{GL}(2,p)$, $(\mathbb{Z}_{p-1}\times\mathbb{Z}_{p-1}):\mathrm{S}_3$, $\mathbb{Z}_{p^2+p+1}:\mathbb{Z}_3$, and $(\mathbb{Z}_3^2:\mathrm{Q}_8).\mathbb{Z}_3$ with $p\equiv 1 (\mathrm{mod}\ 3)$ and $p \not\equiv 1(\mathrm{mod}\ 9)$;

(2) If $M \lessdot \mathrm{PSL}(3,p)$, then M is one of the following groups: $\mathbb{Z}_p^2:\frac{1}{d}\mathrm{GL}(2,p)$, $(\mathbb{Z}_{(p-1)/d}\times\mathbb{Z}_{p-1}):\mathrm{S}_3$, $\mathbb{Z}_{(p^2+p+1)/d}:\mathbb{Z}_3$, $(\mathbb{Z}_3^2:\mathrm{Q}_8).\mathbb{Z}_{\frac{(p-1,9)}{3}}$ with $p\equiv 1(\mathrm{mod}\ 3)$, $\mathrm{PGL}(2,p)$, $\mathrm{PSL}(2,7)$ with $p\equiv 1,2,4(\mathrm{mod}\ 7)$, and A_6 with $p\equiv 1,4(\mathrm{mod}\ 15)$.

- The notation $\frac{1}{d}\mathrm{GL}(2,p)$ is the factor group of $\mathrm{GL}(2,p)$ by the subgroup of order d in $Z(\mathrm{GL}(2,p))$.

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[29] J.N. Bray, D.F. Holt, C.M. Roney-Dougal, The Maximal Subgroups of the Low-Dimensional Finite Classical Groups, Cambridge Univ. Press, 2013.

[30] P.B. Kleidman, M.W. Liebeck, The Subgroup Structure of the Finite Classical Groups, London Mathematical Society Lecture Note Series 129, Cambridge University Press, Cambridge, 1990.

Case 2: $H \cong \mathbb{Z}_p$.

- In the case, $\overline{A_1} \leq \mathrm{PGL}(3,p)$, and $\overline{A_1}$ is 2-transitive on $\overline{S}$.
- If $[\overline{S}]$ is a complete graph. Then $\ell \geq 3(p+1)/2$.
- If $[\overline{S}]$ is a non-complete graph. Then $\ell(\ell-1)(p-1)$ is a divisor of $|\overline{A_1}|$.

Lemma 2.5

Let $p \geq 5$ be a prime. Assume that $X \leq \mathsf{PGL}(3, p)$ has a 2-transitive action on Ω with $n = |\Omega| \geq 4$. Let K be the kernel of X on Ω. If $\mathsf{soc}(X/K)$ is elementary abelian, then either

(1) $(n, X/K) = (4, \mathsf{A}_4)$, $(4, \mathsf{S}_4)$, $(p, \mathsf{AGL}(1, p))$, or $(9, L)$ with $L \leq (\mathbb{Z}_3^2 : \mathsf{Q}_8).\mathbb{Z}_3$ and $1 \equiv (\mathrm{mod}\ 3)$; or

(2) $n = p^2$ and $X/K \leq \mathsf{AGL}(2, p)$.

If $\mathsf{soc}(X/K)$ is nonabelian simple, then one of the following holds:

(1) $(n, X/K) = (5, \mathsf{A}_5)$, $(6, \mathsf{A}_6)$, $(10, \mathsf{A}_6)$, $(8, \mathsf{PSL}(2, 7))$ or $(11, \mathsf{PSL}(2, 11))$, and if $X/K = \mathsf{A}_6$, $\mathsf{PSL}(2, 7)$ or $\mathsf{PSL}(2, 11)$ then $p \equiv 1, 4 (\mathrm{mod}\ 15)$, $p \equiv 1, 2, 4 (\mathrm{mod}\ 7)$ or $p = 11$, respectively;

(2) $n = p + 1$, and $X/K = \mathsf{PSL}(2, p)$ or $\mathsf{PGL}(2, p)$;

(3) $n = p^2 + p + 1$, and $X/K = \mathsf{PSL}(3, p)$ or $\mathsf{PGL}(3, p)$.

Imprimitive Case

- For a vertex-transitive graph Γ and a set of Aut(Γ)-invariant partitions $\mathcal{B}$ of $V(\Gamma)$, the **quotient graph** $\Gamma_{\mathcal{B}}$ of Γ is the graph whose vertex set is the set $\mathcal{B}$ such that two elements $B_i, B_j \in \mathcal{B}$ are adjacent in $\Gamma_{\mathcal{B}}$ if and only if there exist $x \in B_i$ and $y \in B_j$ such that x, y are adjacent in Γ.
- The graph Γ is called a **cover** of $\Gamma_{\mathcal{B}}$ if, for each edge $\{B_i, B_j\}$ of $\Gamma_{\mathcal{B}}$ and $v \in B_i$, the vertex v is adjacent to exactly one vertex in B_j.
- Whenever the blocks in $\mathcal{B}$ are the N-orbits, for some nontrivial normal subgroup N of Aut(Γ), we write $\Gamma_{\mathcal{B}} = \Gamma_N$.

Outline of a proof the main results

[32, Lemma 5.3]

Let $\varGamma$ be a connected locally (G, s)-distance transitive graph with $s \geq 2$. Let $1 \neq N \lhd G$ be intransitive on $V(\varGamma)$, and let $\mathcal{B}$ be the set of N-orbits on $V(\varGamma)$. Then one of the following holds:

(1) $|\mathcal{B}| = 2$;

(2) $\varGamma$ is bipartite, $\varGamma_N \cong \mathsf{K}_{1,r}$ with $r \geq 2$ and G is intransitive on $V(\varGamma)$;

(3) $s = 2$, $\varGamma \cong \mathsf{K}_{m[b]}, \varGamma_N \cong \mathsf{K}_m$, where $m \geq 3$ and $b \geq 2$;

(4) N is semiregular on $V(\varGamma)$, $\varGamma$ is a cover of $\varGamma_N$, $|V(\varGamma_N)| < |V(\varGamma)|$ and $\varGamma_N$ is locally $(G/N, s')$-distance transitive, where $s' = \min\{s, \mathrm{diam}(\varGamma_N)\}$.

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[32] A. Devillers, M. Giudici, C.H. Li, C.E. Praeger, Locally s-distance transitive graphs, J. Graph Theory 69 (2012) 176-197.

Outline of a proof the main results

[32, Proposition 4.2]

Let Γ be a connected locally (G, s)-distance transitive graph with $s \geq 2$. Then there exists no nontrivial $N \lhd G$ such that Γ is a cover of Γ_N and $\Gamma_N \cong \mathsf{K}_{m[b]}$ for some $m \geq 3$ and $b \geq 2$.

Lemma 6.1

For an odd prime p, let Γ be a connected 2-geodesic but not 2-arc transitive graph of order p^2 or p^3, and let $A = \mathrm{Aut}(\Gamma)$. Let $1 \neq N \lhd A$ be intransitive on $V(\Gamma)$ and let Γ be a cover of Γ_N. Then $\Gamma_N \ncong \mathsf{K}_p$.

2-geodesic transitive graph of order p^2: Either $\Gamma \cong \mathsf{K}_{p[p]}$, or Γ is a cover of Σ, where $\Sigma \cong \mathsf{K}_p$.

[32] A. Devillers, M. Giudici, C.H. Li, C.E. Praeger, Locally s-distance transitive graphs, J. Graph Theory 69 (2012) 176-197.

Theorem 4.3

For an odd prime p, let $G = \langle a, b, c \mid a^p = b^p = c^p = [a, c] = [b, c] = 1, c = [a, b] \rangle$, and let $S = \langle b \rangle^* \cup_{i \in \mathbb{Z}_p} \langle b^i a b^i \rangle^*$. Then $\mathrm{Cay}(G, S)$ is 2-geodesic and distance transitive, but not 2-arc transitive, and $\mathrm{Aut}(G, S) \cong \mathrm{GL}(2, p)$. Furthermore,

(1) For every $m \mid (p-1)$, the unique subgroup of index m in $\mathrm{Aut}(E(p^3), S)$ has $2m+2$ orbits: one orbit has length 1, one orbit has length $p^2 - 1$, namely S, m orbits have length $(p-1)/m$, and m orbits have length $(p^2-1)(p-1)/m$;

(2) $\mathrm{Aut}(E(p^3), S)' \cong \mathrm{SL}(2, p)$ has orbit-set $\{\{c^i\}, c^i S \mid i \in \mathbb{Z}_p\}$;

(3) $|SS| = p^2 + (p^2-1)(p-1)$, where $SS = \{s_1 s_2 \mid s_1, s_2 \in S\}$.

2-geodesic transitive graph of order p^3

- Now, A has a nontrivial maximal intransitive normal subgroup N and so A/N is quasiprimitive on $V(\Gamma_N)$.
- Furthermore, either $\Gamma \cong \mathsf{K}_{p^2[p]}$ or $\mathsf{K}_{p[p^2]}$; or N is semiregular on $V(\Gamma)$ and Γ is a cover of Γ_N.
- In particular, $\Gamma_N \cong \mathbf{C}_p, \mathsf{K}_p, \mathbf{C}_{p^2}, \mathsf{K}_{p^2}, H(2,p)$ or $\overline{H(2,p)}$.
- $\Gamma \cong \mathrm{Cay}(M, S)$ and $\Gamma_N \cong \mathrm{Cay}(M/N, SN/N)$, where $M \cong \mathbb{Z}_p^3$ or $\langle a, b, c \mid a^p = b^p = c^p = [a, c] = [b, c] = 1, c = [a, b]\rangle$.
- It is proved that $M \cong \langle a, b, c \mid a^p = b^p = c^p = [a, c] = [b, c] = 1, c = [a, b]\rangle$.
- In this case, $\Gamma_N \ncong \overline{H(2,p)}$. Further, $\Gamma \cong \mathrm{Cay}(M, \langle a\rangle^* \cup \langle b\rangle^*)$ if $\Gamma_N \cong H(2,p)$, and $\Gamma \cong \mathrm{Cay}(M, \langle b\rangle^* \cup_{i\in\mathbb{Z}_p} \langle b^i a b^i\rangle^*)$ if $\Gamma_N \cong \mathsf{K}_{p^2}$.

Thank you!