On finite of 2-arc-transitive Graphs

Speaker: Xiaoqian Zhang

Supervisor: Professor Jing Jian Li

Central South University, Changsha, 410083
Email: zhangxq1110@163.com

Definitions

All graphs here are finite, simple, connected and undirected. Let Γ be a graph with vertex set $V\Gamma$ and edge set $E\Gamma$.

For a vertex $\alpha \in V\Gamma$ and a group G of Γ,

neighborhood: $\Gamma(\alpha) = \{\gamma \in V\Gamma \mid \{\alpha, \gamma\} \in E\Gamma\}$,

vertex-stabilizer: $G_\alpha = \{g \in G \mid \alpha^g = \alpha\}$,

arc-stabilizer: $G_{\alpha\beta} = G_\alpha \cap G_\beta, \beta \in \Gamma(\alpha)$.

Definitions

All graphs here are finite, simple, connected and undirected. Let Γ be a graph with vertex set $V\Gamma$ and edge set $E\Gamma$.

For a vertex $\alpha \in V\Gamma$ and a group G of Γ,

neighborhood: $\Gamma(\alpha) = \{\gamma \in V\Gamma \mid \{\alpha, \gamma\} \in E\Gamma\}$,

vertex-stabilizer: $G_\alpha = \{g \in G \mid \alpha^g = \alpha\}$,

arc-stabilizer: $G_{\alpha\beta} = G_\alpha \cap G_\beta, \beta \in \Gamma(\alpha)$.

An automorphism of Γ is a permutation π on $V\Gamma$ s.t. $\{u, v\} \in E\Gamma \Leftrightarrow \{u^\pi, v^\pi\} \in E\Gamma$.

All automorphisms of Γ form a permutation group on $V\Gamma$, called the full automorphism group of Γ, denoted by $\mathrm{Aut}(\Gamma)$.

Definitions

Frucht theorem (Frucht, Composition Math.1938)

For every abstract finite group G, there is a graph Γ such that $Aut(\Gamma) \cong G$.

Definitions

Frucht theorem (Frucht, Composition Math.1938)

For every abstract finite group G, there is a graph Γ such that $Aut(\Gamma) \cong G$.

The work of Frucht stimulated a great deal of research (groups and graphs).

Definitions

Frucht theorem (Frucht, Composition Math.1938)

For every abstract finite group G, there is a graph Γ such that $Aut(\Gamma) \cong G$.

The work of Frucht stimulated a great deal of research (groups and graphs).

The largest automorphism group is S_n, while the smallest automorphism group is the identity group.

Definitions

Frucht theorem (Frucht, Composition Math.1938)

For every abstract finite group G, there is a graph Γ such that $Aut(\Gamma) \cong G$.

The work of Frucht stimulated a great deal of research (groups and graphs).

The largest automorphism group is S_n, while the smallest automorphism group is the identity group.

Almost all graphs have the trivial automorphism group (see (Blollob*á*s, Random Graphs, 1985)).

Definitions

Let Γ be a graph. An $(s+1)$-sequence $(\alpha_0, \alpha_1, \cdots, \alpha_s)$ of vertices of Γ is called an *s-arc* if $\{\alpha_{i-1}, \alpha_i\} \in E\Gamma$ for $1 \leq i \leq s$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for $1 \leq i \leq s-1$.

Definitions

Let Γ be a graph. An $(s+1)$-sequence $(\alpha_0, \alpha_1, \cdots, \alpha_s)$ of vertices of Γ is called an *s*-arc if $\{\alpha_{i-1}, \alpha_i\} \in E\Gamma$ for $1 \le i \le s$ and $\alpha_{i-1} \ne \alpha_{i+1}$ for $1 \le i \le s-1$.

Denote by $\mathrm{Aut}(\Gamma)$ the full automorphism group of the graph Γ, and a subgroup G of $\mathrm{Aut}(\Gamma)$ is called a group of Γ. A group G of Γ is called

vertex-transitive: G acts transitively on $V\Gamma$,
edge-transitive: G acts transitively on $E\Gamma$,
arc-transitive: G acts transitively on arcs,
s-arc-transitive: G acts transitively on *s*-arcs,

Definitions

Let Γ be a graph. An $(s+1)$-sequence $(\alpha_0, \alpha_1, \cdots, \alpha_s)$ of vertices of Γ is called an s-arc if $\{\alpha_{i-1}, \alpha_i\} \in E\Gamma$ for $1 \leq i \leq s$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for $1 \leq i \leq s-1$.

Denote by $\mathrm{Aut}(\Gamma)$ the full automorphism group of the graph Γ, and a subgroup G of $\mathrm{Aut}(\Gamma)$ is called a group of Γ. A group G of Γ is called

- vertex-transitive: G acts transitively on $V\Gamma$,
- edge-transitive: G acts transitively on $E\Gamma$,
- arc-transitive: G acts transitively on arcs,
- s-arc-transitive: G acts transitively on s-arcs,
- locally transitive: G_α acts transitively on $\Gamma(\alpha)$ for all $\alpha \in V\Gamma$,
- locally primitive: G_α acts primitively on $\Gamma(\alpha)$ for all $\alpha \in V\Gamma$.

Definitions

Normal quotients and covers:

Let Γ be a connected graph, G be locally primitive on Γ, $N \trianglelefteq G$. Suppose that N is intransitive on every G-orbit and has at least 3 orbits on $V\Gamma$. Write $\bar{\alpha} = \alpha^N$ for $\alpha \in V\Gamma$. Let $V\Gamma_N = \{\bar{\alpha} | \alpha \in V\Gamma\}$, and $\bar{G} := G^{V\Gamma_N}$.

Normal quotients and covers:

Let Γ be a connected graph, G be locally primitive on Γ, $N \trianglelefteq G$. Suppose that N is intransitive on every G-orbit and has at least 3 orbits on $V\Gamma$. Write $\bar{\alpha} = \alpha^N$ for $\alpha \in V\Gamma$. Let $V\Gamma_N = \{\bar{\alpha} | \alpha \in V\Gamma\}$, and $\bar{G} := G^{V\Gamma_N}$.

- N is semiregular on $V\Gamma$, $\bar{G} \cong G/N$.
- $G_{\bar{\alpha}} = N \rtimes G_\alpha$, yielding $\bar{G}_{\bar{\alpha}} \cong G_\alpha$.
- $[\bar{\alpha}]_\Gamma$ is empty, $[\bar{\alpha}, \bar{\beta}]_\Gamma$ is either empty or a perfect matching.

Definitions

- $\Gamma_N = (V\Gamma_N, E\Gamma_N)$, $E\Gamma_N = \{\{\bar{\alpha}, \bar{\beta}\} \mid \{\alpha, \beta\} \in E\Gamma\}$, called a normal quotient of Γ. The graph Γ is a cover of Γ_N, called a normal cover of Γ_N.

Definitions

- $\Gamma_N = (V\Gamma_N, E\Gamma_N)$, $E\Gamma_N = \{\{\bar{\alpha}, \bar{\beta}\} \mid \{\alpha, \beta\} \in E\Gamma\}$, called a normal quotient of Γ. The graph Γ is a cover of Γ_N, called a normal cover of Γ_N.

  α and $\bar{\alpha}$ have same valency, and $\bar{G}$ is locally primitive on Γ_N;

  $\bar{G}$ is vertex-transitive iff G is vertex-transitive;

  $\bar{G}$ is 2-arc-transitive iff G is 2-arc-transitive.

Definitions

- $\Gamma_N = (V\Gamma_N, E\Gamma_N)$, $E\Gamma_N = \{\{\bar{\alpha}, \bar{\beta}\} \mid \{\alpha, \beta\} \in E\Gamma\}$, called a normal quotient of Γ. The graph Γ is a cover of Γ_N, called a normal cover of Γ_N.

  α and $\bar{\alpha}$ have same valency, and $\bar{G}$ is locally primitive on Γ_N;

  $\bar{G}$ is vertex-transitive iff G is vertex-transitive;

  $\bar{G}$ is 2-arc-transitive iff G is 2-arc-transitive.

- basic 2-arc-transitive graph: if $G \leqslant \mathrm{Aut}(\Gamma)$ acts transitively on the set of 2-arcs of Γ, and every minimal normal subgroup of G has at most two orbits on $V\Gamma$.

Definitions

- Bipartite graph: A graph Γ is called bipartite graph if Γ is vertex-transitive and its automorphism group has a normal subgroup which has two orbits on $V\Gamma$.

Definitions

- Bipartite graph: A graph Γ is called bipartite graph if Γ is vertex-transitive and its automorphism group has a normal subgroup which has two orbits on $V\Gamma$.
- Let Γ be a bipartite graph and $G \leqslant \mathrm{Aut}(\Gamma)$ is transitive on $V\Gamma$. G has a subgroup G^+ of index 2 with two orbits on $V\Gamma$, say $\triangle_1$ and $\triangle_2$ the two parts of the bipartition of $V\Gamma$.

Definitions

- Bipartite graph: A graph Γ is called bipartite graph if Γ is vertex-transitive and its automorphism group has a normal subgroup which has two orbits on $V\Gamma$.
- Let Γ be a bipartite graph and $G \leqslant \mathrm{Aut}(\Gamma)$ is transitive on $V\Gamma$. G has a subgroup G^+ of index 2 with two orbits on $V\Gamma$, say $\triangle_1$ and $\triangle_2$ the two parts of the bipartition of $V\Gamma$.
- Γ is called G-bi-primitive graph, if G^+ is primitive and faithful on each of $\triangle_1$ and $\triangle_2$, further, if $\mathrm{Aut}(\Gamma) = G$, then Γ is called bi-primitive graph.

Definitions

- bi-Cayley graph: The *bi-Cayley graph* over H with respect to S, denoted by $\mathrm{BiCay}(H, S)$, is defined as a graph whose vertex set is the union of following two copies of H:

$$H_0 = \{h_0 \mid h \in H\}, H_1 = \{h_1 \mid h \in H\},$$

and whose edge set is

$$\{\{h_0, (sh)_1\} \mid h_0 \in H_0, h_1 \in H_1, s \in S\}.$$

Definitions

- bi-Cayley graph: The *bi-Cayley graph* over H with respect to S, denoted by $\mathrm{BiCay}(H, S)$, is defined as a graph whose vertex set is the union of following two copies of H:

$$H_0 = \{h_0 \mid h \in H\}, H_1 = \{h_1 \mid h \in H\},$$

  and whose edge set is

$$\{\{h_0, (sh)_1\} \mid h_0 \in H_0, h_1 \in H_1, s \in S\}.$$

- Normal bi-Cayley graph: If $H \trianglelefteq \mathrm{Aut}(\Gamma)$, then Γ is called a normal bi-Cayley graph of H.

Background

Theorem

Let Γ be an s-arc transitive graph.

- If Γ is cubic, then $s \leqslant 5$ (Tutte 1947);
- If Γ is of valency at least 3, then $s \leqslant 7$ (Weiss 1981).

Background

Theorem

Let Γ be an s-arc transitive graph.

- If Γ is cubic, then $s \leqslant 5$ (Tutte 1947);
- If Γ is of valency at least 3, then $s \leqslant 7$ (Weiss 1981).

This led to one of the central problems in symmetrical graph theory:

Problem

For large values of s, say $s \geqslant 2$,

- characterize s-arc transitive graphs,
- characterize s-arc transitive Cayley graphs.

Background

- Brian Alspach, Marston D.E. Conder, Dragan Marusic, Ming-Yao Xu (1996): Classified 2-arc transitive circulants;
- Dragan Marusic (2003), Shaofei Du, Aleksander Malni, Dragan Marusic (2008): Classified 2-arc transitive dihedrants;
- Li-Xia (2019), Zhou JX (2021): Classified 2-arc transitive solvable Cayley graphs.

Background

- 1-arc transitive Cayley graphs of simple groups with prime valency and solvable stabilizers (Feng，Yin, Zhou, 2021);
- 1-arc transitive Cayley graphs of simple groups with prime valency (Li JJ and Lu ZP, 2021), preprint 48 pages.
- 2-arc transitive Cayley graphs of alternating groups (Pan, Xia, Yin 2021);

Biprimitive 2-arc-transitive bi-Cayley Graphs

Background

- Caiheng Li(2001), Classified the finite vertex primitive and vertex-biprimitive s-transitive graphs for $s \geqslant 4$.

Background

- Caiheng Li(2001), Classified the finite vertex primitive and vertex-biprimitive s-transitive graphs for $s \geqslant 4$.
- Problem: Classify the vertex-primitive and vertex-biprimitive s-transitive graphs for $s = 2$ or 3.

Background

- Caiheng Li(2001), Classified the finite vertex primitive and vertex-biprimitive s-transitive graphs for $s \geqslant 4$.
- Problem: Classify the vertex-primitive and vertex-biprimitive s-transitive graphs for $s = 2$ or 3.
- Caiheng Li (2010), Classified the finite vertex-primitive and vertex-biprimitive s-arc-regular graphs for $s \geqslant 2$.

- Caiheng Li(2001), Classified the finite vertex primitive and vertex-biprimitive s-transitive graphs for $s \geqslant 4$.
- Problem: Classify the vertex-primitive and vertex-biprimitive s-transitive graphs for $s = 2$ or 3.
- Caiheng Li (2010), Classified the finite vertex-primitive and vertex-biprimitive s-arc-regular graphs for $s \geqslant 2$.
- Problem: Classify the vertex-primitive and vertex-biprimitive s-arc-transitive but not s-arc-regular graph for $s \geqslant 2$.

Background

- Caiheng Li and Hua Zhang(2012), Classified the finite vertex-primitive and vertex-biprimitive 2-path-transitive but not 2-arc-transitive graphs.

Background

- Caiheng Li and Hua Zhang(2012), Classified the finite vertex-primitive and vertex-biprimitive 2-path-transitive but not 2-arc-transitive graphs.

- Problem: Classify the vertex-primitive and vertex-biprimitive 2-arc-transitive graphs.

Aim

Classify the biprimitive non-normal 2-arc-transitive bi-Cayley graphs.

Lemmas

Lemma 1. [8, Theorem 2.3]

Let Γ be a finite connected regular bipartite graph with a group G of automorphisms which is 2-arc transitive and bi-quasiprimitive. Then the subgroup G^+ of index 2 in G fixing the parts Δ_1 and Δ_2 of the bipartition setwise has a unique minimal normal subgroup $N \cong T^k$, where $k \geqslant 1$, and T is a simple group, and $(G^+)^{\Delta_1} \cong (G^+)^{\Delta_2}$ is of type I (affine), II (almost simple), III(b)(i) (product action), or III(c) (twisted wreath), as described in [9].

Main Result

Theorem 1

Let Γ is a bi-primitive 2-arc-transitive graph with two parts Δ_1 and Δ_2, $A = \mathrm{Aut}(\Gamma)$ and A^+ be the stabilizer of A on two parts. Then $(A^+)^{\Delta_1} \cong (A^+)^{\Delta_2}$ is of type affine, almost simple or twisted wreath.

Lemma

- Liebeck, Praeger, Saxl's (2010) classification of primitive permutation group containing regular subgroup:

Lemma

- Liebeck, Praeger, Saxl's (2010) classification of primitive permutation group containing regular subgroup:

Lemma 2. [6, Theorem 1.1]

Let G be an almost simple primitive permutation group on a set Ω, with socle L. Suppose that G has a subgroup B which is regular on Ω. Then the possibilities for G, G_α $(\alpha \in \Omega)$ and B are given in Tables 16.1 – 16.3 at the end of the paper.

Strategy

Let $\Gamma = \mathrm{BiCay}(G, S)$ is a biprimitive $(X, 2)$-arc-transitive bi-Cayley graph, where $s \geqslant 2$, $G \leqslant X \leqslant \mathrm{Aut}(\Gamma)$, assume that $G \ntrianglelefteq \mathrm{Aut}(\Gamma)$.

Strategy

Let $\Gamma = \mathrm{BiCay}(G, S)$ is a biprimitive $(X, 2)$-arc-transitive bi-Cayley graph, where $s \geqslant 2$, $G \leqslant X \leqslant \mathrm{Aut}(\Gamma)$, assume that $G \ntrianglelefteq \mathrm{Aut}(\Gamma)$.

- X^+ is an almost simple group.

Strategy

Let $\Gamma = \mathrm{BiCay}(G, S)$ is a biprimitive $(X, 2)$-arc-transitive bi-Cayley graph, where $s \geqslant 2$, $G \leqslant X \leqslant \mathrm{Aut}(\Gamma)$, assume that $G \ntrianglelefteq \mathrm{Aut}(\Gamma)$.

- X^+ is an almost simple group.
- X^+ has a factorization: $X^+ = GX_\alpha^+$, where X_α^+ is the maximal subgroup of X^+ and G is regular on Δ_1 and Δ_2.

Strategy

Let $\Gamma = \mathrm{BiCay}(G, S)$ is a biprimitive $(X, 2)$-arc-transitive bi-Cayley graph, where $s \geqslant 2$, $G \leqslant X \leqslant \mathrm{Aut}(\Gamma)$, assume that $G \ntrianglelefteq \mathrm{Aut}(\Gamma)$.

- X^+ is an almost simple group.
- X^+ has a factorization: $X^+ = GX_\alpha^+$, where X_α^+ is the maximal subgroup of X^+ and G is regular on Δ_1 and Δ_2.
- Determines the structure of $X_\alpha = X_\alpha^+$,
  $X_\alpha = X_\alpha^{[1]}.X_\alpha^{\Gamma(\alpha)} = (X_{\alpha\beta}^{[1]}.(X_\alpha^{[1]})^{\Gamma(\beta)}).X_\alpha^{\Gamma(\alpha)}$.

Condition

Condition

A graph Γ is an $(X,2)$-arc-transitive graph of valency k with $X \leqslant \mathrm{Aut}(\Gamma)$ if and only if there exists a 2-element $g \in X$ such that $\Gamma \cong \mathrm{Cos}(X, X_u, X_u g X_u)$ with $u \in V(\Gamma)$, X_u acts 2-transitively on $[X_u : X_u^g \cap X_u]$ and the following conditions hold:

$$k = |X_u : X_u^g \cap X_u|, g^2 \in X_u, g \in \mathbf{N}_X(X_u^g \cap X_u), \langle g, X_u \rangle = X.$$

Main Result

Let q be a prime power and $n \geq 2$ be an integer. Let $PG(n-1, q)$ be the $(n-1)$-dimensional projective geometry over a field of order q. Let V_1 and V_{n-1} be the sets of points and hyperplanes of $PG(n-1, q)$, respectively.

Main Result

Let q be a prime power and $n \geq 2$ be an integer. Let $PG(n-1, q)$ be the $(n-1)$-dimensional projective geometry over a field of order q. Let V_1 and V_{n-1} be the sets of points and hyperplanes of $PG(n-1, q)$, respectively.

- The *point-hyperplane incidence graph*, denoted by $HP(n-1, q)$, of $PG(n-1, q)$ is the graph with vertices $V_1 \cup V_{n-1}$, and $\alpha \in V_1$ and $\beta \in V_{n-1}$ are adjacent if and only if $\alpha \subseteq \beta$.

Main Result

Let q be a prime power and $n \geq 2$ be an integer. Let $PG(n-1, q)$ be the $(n-1)$-dimensional projective geometry over a field of order q. Let V_1 and V_{n-1} be the sets of points and hyperplanes of $PG(n-1, q)$, respectively.

- The *point-hyperplane incidence graph*, denoted by $HP(n-1, q)$, of $PG(n-1, q)$ is the graph with vertices $V_1 \cup V_{n-1}$, and $\alpha \in V_1$ and $\beta \in V_{n-1}$ are adjacent if and only if $\alpha \subseteq \beta$.

- The *point-hyperplane non-incidence graph*, denoted by $\overline{HP(n-1, q)}$, of $PG(n-1, q)$ is the graph with vertices $V_1 \cup V_{n-1}$, and $\alpha \in V_1$ and $\beta \in V_{n-1}$ are adjacent if and only if $\alpha \nsubseteq \beta$.

Main Result

Theorem 2

Let Γ be a bi-primitive s-arc-transitive bi-Cayley graph over a group B with $s \geq 2$. Let $A = \mathrm{Aut}(\Gamma)$. Then one of the following holds:

(a) Γ is a normal bi-Cayley graph;

(b) Γ is the complete bipartite graph $\mathrm{K}_{n,n}$;

(c) $\Gamma \cong \Delta^{(2)}$, where $\Delta = \mathrm{K}_n$ with $n \geq 3$, or Δ is one of the graphs given in [5, Table 1];

(d) $\Gamma \cong HP(n-1, q)$, $\overline{HP(n-1, q)}$ $(n \geq 3)$, $G(22, 5)$ or $B'(H(11))$ (see [7] for the last two graphs);

(e) Γ is one of the graphs given in *Examples 4.3, 4.4, 4.5, 4.11, 4.13 and 4.14*.

Basic 2-arc-transitive graphs of order $r^a s^s$ or $2r^a s^s$

Background

- C.E. Praeger (1993) proved that a connected 2-arc-transitive graph is a cover of some basic 2-arc-transitive graph.

Background

- C.E. Praeger (1993) proved that a connected 2-arc-transitive graph is a cover of some basic 2-arc-transitive graph.
- **Problem:** Classify all finite basic 2-arc-transitive graphs.

Background

- C.E. Praeger (1993) proved that a connected 2-arc-transitive graph is a cover of some basic 2-arc-transitive graph.
- **Problem:** Classify all finite basic 2-arc-transitive graphs.

Lemma 3. [9, Theorem 2.]

Let Γ be a finite connected graph with a group G of automorphisms which is 2-arc transitive on Γ and quasiprimitive on the vertex set Ω of Γ. Then G is of HA, AS, PA or TW as described in [9].

Background

- C.E. Praeger (1993) proved that a connected 2-arc-transitive graph is a cover of some basic 2-arc-transitive graph.
- **Problem:** Classify all finite basic 2-arc-transitive graphs.

Lemma 3. [9, Theorem 2.]

Let Γ be a finite connected graph with a group G of automorphisms which is 2-arc transitive on Γ and quasiprimitive on the vertex set Ω of Γ. Then G is of HA, AS, PA or TW as described in [9].

- C.E. Praeger (1993) proved that if Γ is bipartite, then either Γ is a complete bipartite graph, or G^+ is faithful on both parts of Γ.

Background

- Caiheng Li (2001), classified the finite s-arc-transitive graphs of prime-power order.

Background

- Caiheng Li (2001), classified the finite s-arc-transitive graphs of prime-power order.
- Caiheng Li, Zaiping Lu and Jingjian Li (2010): Classified 2-arc-transitive graph of odd order.

Background

- Caiheng Li (2001), classified the finite s-arc-transitive graphs of prime-power order.
- Caiheng Li, Zaiping Lu and Jingjian Li (2010): Classified 2-arc-transitive graph of odd order.
- Luke Morgan. et al. (2016), for any given positive integers k, n and d with d large enough, there exist only a finite number of primes r such that kr^n is the order of some 2-arc-transitive graph of valency d.

Aim

Classify the basic 2-arc-transitive graphs of order $r^a s^b$ and $2r^a s^b$ (bipartite).

Lemmas

- Caiheng Li, Xianhua Li's (2014) classification of permutation groups of degree a product of two prime-powers:

Lemmas

- Caiheng Li, Xianhua Li's (2014) classification of permutation groups of degree a product of two prime-powers:

Lemma 4. [1, Theorem 1.2]

Let T be a finite simple group and $X < T$ with $|T : X| = p^a q^b$, where p, q are primes and a, b are positive integers, then X is isomorphic to K or H and (T, X) lies in Tables 3.2, 3.1, 4.1, 4.2, 4.3, 4.4, 4.5 and 5.1.

Main Result

Theorem 3

Let a be a positive integer, r and s be distinct primes. Assume that $\Gamma = (V, E)$ is a basic 2-arc-transitive graph with respect to a group G. Let $\{\alpha, \beta\} \in E$ and $G^+ = \langle G_\alpha, G_\beta \rangle$. Assume that G is almost simple with $\mathrm{soc}(G) = T$ and G^+ has an orbit on V of length $r^a s^b$. Then Γ is isomorphic to one of the following graphs:

(1) the complete graph $\mathrm{K}_{r^a s^b}$ and its standard double cover;

(2) the Odd graph O_4 of valency 4 and its standard double cover;

(3) the point-hyperplane incidence graph and non-incidence graph of the projective geometry $\mathrm{PG}(n-1, q)$, where $n \geqslant 3$ and $r^a s^b = \frac{q^n - 1}{q - 1}$;

(4) the incidence graph of the generalized quadrangle $\mathrm{GQ}(4, 2^{2^i})$, where $i \geqslant 1$;

(5) the graphs in Examples 3.1, 3.2, 3.4-3.10, the standard double cover of the graph in Example 3.1, and the graphs described as in Tables 4.2 and 6.7.

(6) $T = \mathrm{PSL}_2(p^{2^i})$, $T_\alpha = \mathrm{C}_p^{2^i} \rtimes \mathrm{C}_m$, and Γ is of valency p^{2^i} and described as in [4, Table 1], where p is an odd prime and $m = \frac{p^{2^i} - 1}{2^{i_0+1}}$ for $1 \leqslant i_o < i$.

On locally primitive symmetric graph of valency twice prime

Background

- Giudici et al. (2003) established a reduction for studying locally primitive bipartite graphs.

Background

- Giudici et al. (2003) established a reduction for studying locally primitive bipartite graphs.
- Caiheng Li, Jiangmin Pan and Li Ma (2009) gave a characterization of the undirected vertex transitive locally primitive graphs of order prime power.

Background

- Giudici et al. (2003) established a reduction for studying locally primitive bipartite graphs.
- Caiheng Li, Jiangmin Pan and Li Ma (2009) gave a characterization of the undirected vertex transitive locally primitive graphs of order prime power.
- Xingui Fang, Xuesong Ma, Jie Wang (2011): the locally-primitive Cayley graphs of finite non- abelian simple groups are investigated.

Background

- Giudici et al. (2003) established a reduction for studying locally primitive bipartite graphs.
- Caiheng Li, Jiangmin Pan and Li Ma (2009) gave a characterization of the undirected vertex transitive locally primitive graphs of order prime power.
- Xingui Fang, Xuesong Ma, Jie Wang (2011): the locally-primitive Cayley graphs of finite non- abelian simple groups are investigated.
- Jiangmin Pan (2014) classify locally-primitive Cayley graphs of dihedral groups.

Aim

Classify the locally primitive symmetric graphs of valency $2p$ admitting a transitive alternating automorphism group.

Lemmas

Lemma 5.

Let Γ be a G-locally primitive graph with $H < G \leqslant \mathrm{Aut}(\Gamma)$ such $H \cong \mathrm{A}_n$ is transitive on $V\Gamma$, where $n \geqslant 5$. Take N an arbitrary normal subgroup of G. Then either G acts quasi-primitively on $V\Gamma$ or N has at least three orbits on $V\Gamma$ and the normal quotient Γ_N is G/N-locally primitive, in particular, Γ is a normal cover of Γ_N for the latter.

Lemmas

Lemma 6.

Let Γ be a G-locally primitive graph of valency $2r$, where r is an odd prime and $G \leqslant \mathrm{Aut}\Gamma$ acts quasi-primitively on $V\Gamma$. If G contains a proper transitive subgroup isomorphic to an alternating simple group, then either Γ is a normal Cayley graph of A_n or G is an almost simple group.

Lemmas

- Binzhou Xia (2017) classified quasi-primitive permutation groups with a transitive subgroup which is isomorphic to A_n for some $n \geqslant 5$:

Lemmas

- Binzhou Xia (2017) classified quasi-primitive permutation groups with a transitive subgroup which is isomorphic to A_n for some $n \geqslant 5$:

Lemma 7. [10, Theorem 1.1]

Suppose that $L = HK$ is a factorization of the finite simple group L with $H \cong \mathrm{A}_n$, where $n \geqslant 5$. Then one of the following holds.

(a) $L = \mathrm{A}_{n+k}$ for $1 \leqslant k \leqslant 5$, and L_α is k-transitive on $n+k$ points.

(b) $L = \mathrm{A}_m$ and $L_\alpha = \mathrm{A}_{m-1}$, where m is the index of a subgroup in A_n.

(c) (L, n, L_α) lies in [10, Table 1].

Main Result

Theorem 4.

Let Γ be a G-symmetric graphs of valency $2r$, where $G \leqslant \mathrm{Aut}\Gamma$ and r is an odd prime. Assume that Γ is G-locally primitive and $H \cong \mathrm{A}_n$ acts transitively on $V\Gamma$ for $H < G$ and $n \geqslant 5$ integer. Then either Γ is isomorphic to a normal Cayley graph of A_n, or G is almost simple with socle L and one of the following holds.

(1) Γ is isomorphic to the complete graph K_{2r+1};

(2) $(L, H) = (\mathrm{A}_{n+1}, \mathrm{A}_n)$ and L_α is 1-transitive but not 2-transitive on $n+1$ points;

(3) $(L, L_\alpha^{\Gamma(\alpha)}, r)$ is $(\mathrm{A}_{q+1}, \mathrm{PSL}(2,q).\mathcal{O}, \frac{q+1}{2})$ and $H \cong \mathrm{A}_{q-1}$, where $\mathcal{O} \leqslant \mathrm{Out}(\mathrm{PSL}(2,q))$, $q = p^f$ for p some odd prime and f a power of 2;

(4) Γ is a normal cover of one of the above graphs.

Reference

[1] C. H. Li, X.H. Li. On permutation groups of degree a product of two prime-powers [J], Communications in Algebra, 2014, 42(4): 721–4743.

[2] X.G. Fang and C.E. Praeger, Finite two-arc-transitive graphs admitting a Suzuki simple group [J], Communications in Algebra, 1999, 27(8): 3727-3754.

[3] X.G. Fang and C.E. Praeger, Finite two-arc-transitive graphs admitting a Ree simple group [J], Communications in Algebra, 1999, 27(8): 3755-3769.

[4] A. Hassani, L.R. Nochefranca, C.E. Praeger, Two-arc transitive graphs admitting a two-dimensional projective linear group [J], Journal of Group Theory, 1999, 2(4): 335-353.

[5] J. J. Li, J. Yang and W. Y. Zhu, Vertex primitive s-transitive Cayley graphs, *Discrete Mathematics*, **343** (2020).

[6] M.W. Liebeck, C.E. Praeger and J. Saxl, Regular subgroups of primitive permutation groups, *Mem. Amer. Math. Soc.*, 2010.

[7] Y. Cheng and J. Oxley, On weakly symmetric graphs of order twice a prime, *J. Comb. Theory, Ser. B*, **42** (1987), 196–211.

[8] C.E. Praeger, On a reduction theorem for finite, bipartite 2-arc-transitive graphs, *Australas. J. Combin*, **7** (1993), 21–36.

[9] C.E. Praeger, An O'Nan-Scott Theorem for Finite Quasiprimitive Permutation Groups and an Application to 2-Arc Transitive Graphs. *Journal of the London Mathematical Society*, **2** (1993), 227–239.

[10] B. Xia, Quasiprimitive groups containing a transitive alternating group. *Journal of Algebra*, **490** (2017), 555-567.

Thank You!