Finite Locally Primitive 2-designs

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June 21, 2022

Definition

Define $\mathscr{D} = (\mathscr{P}, \mathscr{B})$ as a $2-(v,k,\lambda)$ design, if

(1) $\mathscr{P}$ is a set of v points.

(2) $\mathscr{B}$ is a set of blocks, where each block is a $k-$subset of $\mathscr{P}$.

(3) each two distinct points are contained in exactly λ blocks.

For any $\alpha \in \mathscr{P}$ and $B \in \mathscr{B}$, if $\alpha \in B$, we say (α, B) is a flag. Define b as the number of blocks, and r as the number of blocks contain α. Then we have

(1) $vr = bk$.

(2) $\lambda(v-1) = r(k-1)$.

Definition

An automorphism of $\mathscr{D}$ is a permutaiton of $\mathscr{P}$ which preserves the block set $\mathscr{B}$, i.e. $Aut(\mathscr{D}) = \{g \in Sym(\mathscr{P}) | B^g \in \mathscr{B}, \forall B \in \mathscr{B}\}$.

Lemma

Let $\mathscr{D} = (\mathscr{P}, \mathscr{B})$ be a $2-(v,k,\lambda)$ design. Let $G \leqslant Aut(\mathscr{D})$. Then G acts faithfully on $\mathscr{P}$, on $\mathscr{B}$ and on flags.

Proof.

(1) As $G \leqslant Aut(\mathscr{D}) \leqslant Sym(\mathscr{P})$, G acts faithfully on $\mathscr{P}$.

(2) For any $g \in G$, if g fixes all blocks and $g \neq 1$, then there exist α_1 and α_2 in $\mathscr{P}$, such that $\alpha_1^g = \alpha_2$. For all $B \in \Gamma(\alpha_1)$, $B^g = B$, and $\alpha_2 = \alpha_1^g \in B^g = B$. Thus $\{\alpha_1, \alpha_2\}$ are on all r blocks contain α_1. But $r \neq \lambda$. A contradiction.

(3) For any $g \in G$, if g fixes all flags, then g is also fixes all points, i.e. $g = 1$.

□

Let $G \leqslant Aut(\mathscr{D})$. We say

(1) G is flag-transitive, if G acts transitively on the set of flags of $\mathscr{D}$.

(2) G is point-(quasi)primitive, if G acts (quasi)primitively on $\mathscr{P}$.

(3) G is block-(quasi)primitive, if G acts (quasi)primitively on $\mathscr{B}$.

(4) G is locally-primitive, if G_α is primitive on $\Gamma(\alpha) = \{B \in \mathscr{B} | \alpha \in B\}$ and G_B is primitive on $\Gamma(B) = \{\alpha \in \mathscr{P} | \alpha \in B\}$.

Lemma

Let $\mathscr{D}$ be a $G-$locally primitive $2-$design. Then $\mathscr{D}$ is $G-$point-primitive.

Problem

Classify locally primitive 2-design.

Lemma (Li,2003,Analysing finite locally s-arc transitive graph)

Let Γ be a finite G-locally primitive connected graph such that G has two orbits on vertices and G acts faithfully and quasiprimitive on both orbits with type $\{X,Y\}$. Then either $X{=}Y$, or $\{X,Y\}{=}\{PA,SD\}$ or $\{PA,CD\}$, and examples exist in each case.

Lemma (Li,2003,Analysing finite locally s-arc transitive graph)

Let Γ be a connected G-locally quasiprimitive graph. Suppose that Γ is bipartite and orbits of G are the bipartite halves Δ_1 and Δ_2. Suppose also that G acts faithfully and quasiprimitive on Δ_1 and Δ_2. Let $N \lhd G$.

(1) N^{Δ_1} *is regular* $\iff$ N^{Δ_2} *is regular.*

(2) *If N is not regular on Δ_1, then $N_v^{\Gamma(v)}$ is transitive for all $v \in V\Gamma$.*

Suppose $\mathscr{D} = (\mathscr{P}, \mathscr{B})$ is a G-locally-primitive $2-(v,k,\lambda)$ design. Then $\mathscr{D}$ is G-point-primitive. Suppose also that $\mathscr{D}$ is G-block-quasiprimitive. Then either $(G^{\mathscr{P}}, G^{\mathscr{B}}) = (HA, HA), (TW, TW)$, or $\mathscr{D}$ is $soc(G)$-flag-transitive.

Lemma

Let $\mathscr{D} = (\mathscr{P}, \mathscr{B})$ be a G-locally-transitive $2-(v,k,\lambda)$ design. Then $G^{\mathscr{P}}$ is not of type SD,CD,HS,HC.

Proof.

Suppose $G^{\mathscr{P}}$ is of type SD,CD,HS,HC. Let $N = soc(G) \cong T^m$. Then N is not regular on $\mathscr{P}$, thus N_α is transitive on $\Gamma(\alpha)$. Suppose $N_\alpha \cong T^i$. Then $v = \frac{|N|}{|N_\alpha|} = |T|^{m-i}$ and $r = \frac{|N_\alpha|}{|N_{\alpha B}|} \,\big|\, |N_\alpha| = |T|^i$, and $gcd(v-1, r) \mid gcd(|T|^{m-i} - 1, |T|^i) = 1$. As $\lambda(v-1) = r(k-1), r \mid \lambda$. But $\lambda < r$. A contradiction. □

Lemma

Let $\mathscr{D} = (\mathscr{P}, \mathscr{B})$ be a G-point-primitive $2-(v,k,\lambda)$ design. Then $G^{\mathscr{P}}$ is not of type TW.

Lemma

Let $\mathscr{D}=(\mathscr{P},\mathscr{B})$ *be a* G*-flag-transitive* $2-(v,k,\lambda)$ *design. Let* (α,B) *be a flag, where* $\alpha\in\mathscr{P}$ *and* $B\in\mathscr{B}$. *Let* $\mathscr{D}'=(\mathscr{P}',\mathscr{B}')$ *be a design, where* $\mathscr{P}'=\{G_\alpha g|g\in G\},\mathscr{B}'=\{G_\alpha G_B g|g\in G\}$ *and* $(G_\alpha g_1,G_\alpha G_B g_2)$ *is a flag if and only if* $G_\alpha g_1\ \subset G_\alpha G_B g_2$. *Then*

(1) $\mathscr{D}\cong\mathscr{D}'$.

(2) $\lambda=\frac{|G_B G_\alpha\cap G_B G_\alpha g|}{|G_B|}$, *for all* $g\notin G_\alpha$.

(3) $\lambda=\frac{|G_B G_\alpha\cap G_B G_{\alpha'}|}{|G_B|}$, *where* (α,B) *and* (α',B) *are two flags.*

Proof.

(1) Define $f: \mathscr{P} \to \mathscr{P}'$ be a map, where $f: \alpha^g \mapsto G_\alpha g$. Then $\forall g_1, g_2 \in G$,

$$\alpha^{g_1} = \alpha^{g_2} \iff \alpha^{g_1 g_2^{-1}} = \alpha \iff g_1 g_2^{-1} \in G_\alpha \iff G_\alpha g_1 = G_\alpha g_2.$$

Thus $f: \mathscr{P} \to \mathscr{P}'$ is a bijection.
As $\mathscr{D}$ is a G-flag-transitive design, $B = \{\alpha^g | g \in G_B\} = \alpha^{G_B}$,
and $f(B) = \{f(\alpha^h) | h \in G_B\} = \{G_\alpha h | h \in G_B\} = G_\alpha G_B$.
Then $f(B^g) = \{f(\alpha^{hg}) | h \in G_B\} = \{G_\alpha hg | h \in G_B\} = G_\alpha G_B g, \forall g \in G$.
For all $g_1, g_2 \in G$,

$$B^{g_1} = B^{g_2} \implies B^{g_1 g_2^{-1}} = B \implies g_1 g_2^{-1} \in G_B \implies G_B g_1 = G_B g_2$$

$$\implies G_\alpha G_B g_1 = G_\alpha G_B g_2 \implies \alpha^{G_\alpha G_B g_1} = \alpha^{G_\alpha G_B g_2} \implies B^{g_1} = B^{g_2}.$$

Thus $f: \mathscr{B} \to \mathscr{B}'$ is a bijection, and $\mathscr{D} \cong \mathscr{D}'$.

□

Proof.

(2) For all $g_1, g_2 \in G$,

$$\alpha^{g_1} \in B^{g_2} \iff G_\alpha g_1 \subset G_\alpha G_B g_2 \iff G_\alpha \subset G_\alpha G_B g_2 g_1^{-1}$$
$$\iff G_\alpha \cap G_B g_2 g_1^{-1} \neq \emptyset \iff G_\alpha g_1 \cap G_B g_2 \neq \emptyset \iff G_\alpha g_1 g_2^{-1} \cap G_B \neq \emptyset$$
$$\iff G_B \subset G_B G_\alpha g_1 g_2^{-1} \iff G_B g_2 \subset G_B G_\alpha g_1.$$

Let B^g be a block contains two distinct points $\alpha^{g_1}, \alpha^{g_2}$.
Then $\{\alpha^{g_1}, \alpha^{g_2}\} \subset B^g \iff G_B g \subset G_B G_\alpha g_1 \cap G_B G_\alpha g_2$.
The number of blocks contain $\alpha^{g_1}, \alpha^{g_2}$, say $\lambda_{\alpha^{g_1}, \alpha^{g_2}}$, is equal to the number of G_B cosets on $G_B G_\alpha g_1 \cap G_B G_\alpha g_2$,i.e.

$$\lambda_{\alpha^{g_1}, \alpha^{g_2}} = \frac{|G_B G_\alpha g_1 \cap G_B G_\alpha g_2|}{|G_B|}.$$

Thus

$$\lambda = \frac{|G_B G_\alpha \cap G_B G_\alpha g|}{|G_B|},$$

where $g \notin G_\alpha$.

Lemma

Let $\mathscr{D} = (\mathscr{P}, \mathscr{B})$ be a G-locally-primitive $2-(v,k,\lambda)$ design.
Suppose G is primitive on $\mathscr{P}$ of type PA, and G is quasiprimitive on $\mathscr{B}$.
Let $N = soc(G) = T_1 \times T_2 \times \cdots T_m \cong T^m$.
Let (α, B) be a flag, where $\mathscr{P} = \Omega^m, \omega_i \in \Omega, \alpha = (\omega_1, \omega_2, \cdots, \omega_m), B \in \mathscr{B}$.
Let π be a non-trivial subset of $[m] = \{1, 2, \cdots, m\}$.
Let $\pi' = [m] \setminus \pi$.
Define $\rho_\pi : N \to N^\pi \cong T^{|\pi|}$ by $(n_1, n_2, \cdots, n_m) \mapsto (h_1, h_2, \cdots, h_m)$, where $h_i = n_i$ if $i \in \pi$, and $h_i = 1$ if $i \notin \pi$.
Then $N_B \neq N_B^\pi \times N_B^{\pi'}$, for all non-trivial subset π of $[m]$.

Proof.

Suppose there exist a non-trivial subset π of $[m]$, such that $N_B = N_B^{\pi} \times N_B^{\pi'}$. Then $N = N^{\pi} \times N^{\pi'}$, and $N_\alpha = N_\alpha^{\pi} \times N_\alpha^{\pi'}$.
As $\mathscr{D}$ is $N-$flag-transitive,

$$\lambda = \frac{|N_B N_\alpha \cap N_B N_\alpha n|}{|N_B|}, \forall n \notin N_\alpha.$$

Let $x^{\pi} \in N^{\pi} \setminus N_\alpha^{\pi}$. Let $x^{\pi'} \in N^{\pi'} \setminus N_\alpha^{\pi'}$. Then $x^{\pi} \cdot x^{\pi'}, 1 \cdot x^{\pi'} \notin N_\alpha$. Thus

$$|N_B N_\alpha \cap N_B N_\alpha (x^{\pi} \cdot x^{\pi'})| = |N_B N_\alpha \cap N_B N_\alpha (1 \cdot x^{\pi'})|,$$

which implies

$$|N_B^{\pi} N_\alpha^{\pi} \cap N_B^{\pi} N_\alpha^{\pi} x^{\pi}| \cdot |N_B^{\pi'} N_\alpha^{\pi'} \cap N_B^{\pi'} N_\alpha^{\pi'} x^{\pi'}| = |N_B^{\pi} N_\alpha^{\pi}| \cdot |N_B^{\pi'} N_\alpha^{\pi'} \cap N_B^{\pi'} N_\alpha^{\pi'} x^{\pi'}|$$

Proof.

Then

$$|N_B^\pi N_\alpha^\pi \cap N_B^\pi N_\alpha^\pi x^\pi| = |N_B^\pi N_\alpha^\pi|.$$

And

$$N_B^\pi N_\alpha^\pi x^\pi = N_B^\pi N_\alpha^\pi,$$

for all $x^\pi \in N^\pi \setminus N_\alpha^\pi$, which implies

$$N_B^\pi N_\alpha^\pi = \cup_{x^\pi \in N^\pi} N_B^\pi N_\alpha^\pi x^\pi = N_B^\pi N^\pi = N^\pi.$$

Similarly,

$$N_B^{\pi'} N_\alpha^{\pi'} = N^{\pi'}.$$

Thus

$$N_B N_\alpha = (N_B^\pi N_\alpha^\pi) \times (N_B^{\pi'} N_\alpha^{\pi'}) = N^\pi \times N^{\pi'} = N$$

and $\lambda = \frac{|N|}{|N_B|} = b$. A contradiction. □

Suppose G is primitive on $\mathscr{P}$ of type PA, and G is quasiprimitive on $\mathscr{B}$. Then $(G^{\mathscr{P}}, G^{\mathscr{B}}) \neq (PA, CD)$.
If $(G^{\mathscr{P}}, G^{\mathscr{B}}) = (PA, PA)$, then G is not primitive on $\mathscr{B}$.
(Suppose $G^{\mathscr{B}}$ is quasiprimitive of type PA. Then N_B is a subdirect product of R^m for some $1 < R < T$.)

Lemma

Let $\mathscr{D} = (\mathscr{P}, \mathscr{B})$ be a G-locally-primitive $2-(v,k,\lambda)$ design. And G is not primitive on $\mathscr{B}$. Let $\mathscr{S} = \{S_1, S_2, \cdots\}$ be a nontrivial G-invariant partition of $\mathscr{B}$, i.e. $S_i^g = S_j, \forall g \in G$. Let (α, B) be a flag and $B \in S \in \mathscr{S}$. Define $G_S = \{g \in G | B^g \in S\}$. Then

(1) $G_\alpha \cap G_B = G_\alpha \cap G_S$.
(2) *For any two block B_1, B_2 in S, $B_1 \cap B_2 = \emptyset$.*

Lemma

Let $\mathscr{D}=(\mathscr{P},\mathscr{B})$ be a G-locally-primitive $2-(v,k,\lambda)$ design. Then $(G^{\mathscr{P}},G^{\mathscr{B}})\neq(PA,SD)$.

Define $N=soc(G)$. Then $\mathscr{D}$ is $N-$flag-transitive.
As $G^{\mathscr{B}}$ is of type SD, we have $T^m\lhd\ G\leqslant A\rtimes S_n$, where
$A=\{(a_1,a_2,\cdots,a_m)|a_i\in Aut(T), Ta_i=Ta_j,\forall i,j\}$.
Let (α,B) be a flag, where $\alpha=(\omega_1,\omega_2,\cdots,\omega_m)\in\mathscr{P}$, and $B\in\mathscr{B}$. Then $N_\alpha=\{(t_1,t_2,\cdots,t_m)|t_i\in T_{\omega_i}\}\cong T_{\omega_1}\times T_{\omega_2}\times\cdots T_{\omega_m}$, and without loss of generality, let $N_B=\{(t,t,\cdots,t)|t\in T\}\cong T$.
Define $D:Aut(T)\to D(Aut(T))=\{(a,a,\cdots,a)|a\in Aut(T)\}\leqslant Aut(T)^m$ by $a\mapsto(a,a,\cdots,a)$, for $a\in Aut(T)$.
Define $\rho:D(Aut(T))\times S_m\to Aut(T)$ by $(a,a,\cdots,a)s\mapsto a$ for $a\in Aut(T)$ and $s\in S_m$.
Then $G_B\leqslant D(Aut(T))\times S_m$.

Define $n=(n_1,1,\cdots,1)\in N$, where $n_1\notin T_{\omega_1}$. Then $n\notin N_\alpha$ and $\alpha^n\neq\alpha$. The number of blocks contain both α and α^n is

$$\lambda_{\alpha,\alpha^n}=\frac{|N_BN_\alpha\cap N_BN_\alpha n|}{|N_B|}.$$

Suppose $N_B(t_1,t_2,\cdots,t_m)\subseteq N_BN_\alpha\cap N_BN_\alpha n$, where $(t_1,t_2,\cdots,t_m)\in N_\alpha$, then there exist $(t_1',t_2',\cdots,t_m')\in N_\alpha$, such that

$$N_B(t_1,t_2,\cdots,t_m)=N_B(t_1',t_2',\cdots,t_m')n,$$

i.e.$(t_1'nt_1^{-1},t_2't_2^{-1},\cdots,t_m't_m^{-1})\in N_B$, where $t_i,t_i'\in T_{\omega_i}$. There exist $t\in T$ such that $t=t_1'n_1t_1^{-1}=t_2't_2^{-1}=\cdots=t_m't_m^{-1}$.
Thus $t\in T_{\omega_2}\cap\cdots\cap T_{\omega_m}, t_1=t^{-1}t_1'n_1\in(T_{\omega_2}\cap\cdots\cap T_{\omega_m})T_{\omega_1}n_1\cap T_{\omega_1}$.
Then $T_{\omega_2}\cap\cdots\cap T_{\omega_m}\subsetneq T_{\omega_1}$, otherwise $t_1\in T_{\omega_1}n_1\cap T_{\omega_1}=\emptyset$.
Thus there exist $x\in T_{\omega_2}\cap\cdots\cap T_{\omega_m}$ where $x\notin T_{\omega_1}$ and

$$T_{\omega_1}\cap T_{\omega_2}\cap\cdots\cap T_{\omega_m}<T_{\omega_2}\cap\cdots\cap T_{\omega_m}.$$

These implies $\omega_1\neq\omega_i,\forall i\neq 1$. Similarly, $\omega_j\neq\omega_i,\forall i\neq j$. And $\{\omega_1,\omega_2,\cdots,\omega_m\}$ are distinct elements. Thus $m\leqslant|\Omega|$. Suppose $m=|\Omega|$, then $T_{\omega_2}\cap\cdots\cap T_{\omega_m}$ fixes ω_1. A contradiction. Then $m<|\Omega|$.

As $G_{\alpha B} \leqslant G_B \leqslant D(Aut(T)) \times S_m$,
for all $g \in G_{\alpha B}, g = (a, a, \cdots, a)s$, forsome $a \in Aut(T), s \in S_m$.
$N_\alpha^g = N_\alpha$, i.e.

$$\begin{aligned} N_\alpha^g &= N_\alpha^{(a,\cdots,a)s} \\ &= \{(t_1, t_2, \cdots, t_m) | t_i \in T_{\omega_i}\}^{(a,\cdots,a)s} \\ &= \{(t_1, t_2, \cdots, t_m) | t_i \in T_{\omega_i^a}\}^s \\ &= N_\alpha. \end{aligned}$$

Thus

$$\{T_{\omega_1}, \cdots, T_{\omega_m}\} = \{T_{\omega_1^a}, \cdots, T_{\omega_m^a}\}.$$

Define $Y = T_{\omega_1} \cap \cdots \cap T_{\omega_m}$, and $Y^m = \{(y_1, y_2, \cdots, y_m) | y_i \in Y\}$. Then

$$Y^a = T_{\omega_1^a} \cap \cdots \cap T_{\omega_m^a} = T_{\omega_1} \cap \cdots \cap T_{\omega_m} = Y.$$

As S_m acts on $\{1, 2, \cdots, m\}$ faithfully, there exist unique $s \in S_m$, such that $(a, a, \cdots, a)s \in G_{\alpha B}$. Thus $\rho|_{G_{\alpha B}}$ is an injection.
$|G_{\alpha B}| = |\rho(G_{\alpha B})| \leqslant |Aut(T)|$.

(1) Suppose $Y \neq 1$.
Then $N_{\alpha B} < Y^m < N_\alpha$. And
$G_{\alpha B} = N_{\alpha B} G_{\alpha B} \subset Y^m G_{\alpha B} \subset N_\alpha G_{\alpha B} = G_\alpha$.
For all $g = (a, \cdots, a)s \in G_{\alpha B}$, where $a \in Aut(T)$ and $s \in S_m$,

$$(Y^m)^g = (Y \times \cdots \times Y)^{(a,\cdots,a)s} = (Y^a \times \cdots \times Y^a)^s = Y \times \cdots \times Y = Y^m.$$

Thus $Y^m G_{\alpha B}$ is a group, and $Y^m \lhd Y^m G_{\alpha B}$.
Let $y = (y_1, y_2, \cdots, y_m) \in Y^m$, where $y_1 \neq y_2$.
Then $y \notin G_{\alpha B}$, and $G_{\alpha B} \neq Y^m G_{\alpha B}$.
As $Y^m < N$, we obtain $Y^m G_{\alpha B} \cap N = Y^m (G_{\alpha B} \cap N) = Y^m N_{\alpha B} = Y^m$.
But $G_\alpha \cap N = N_\alpha \neq Y^m$. Thus $Y^m G_{\alpha B} \neq G_\alpha$.
Then $G_{\alpha B} < Y^m G_{\alpha B} < G_\alpha$, and $G_{\alpha B}$ is not a maximal subgroup of G_α. A contradiction.

(2) Suppose $Y = 1$.
Then $N_{\alpha B} = \{(t, \cdots, t) | t \in Y\} = 1$.
As $\rho : G_B \to Aut(T)$ is a group homomorphism, $\rho(G_{\alpha B}) \leqslant \rho(G_B)$.

(2.1) Suppose $\rho(G_{\alpha B}) = \rho(G_B)$.
As $\rho(N_B) = T \leqslant \rho(G_B)$, and T is transitive on Ω , $m < |\Omega|$, there exist $t \in T \leqslant \rho(G_{\alpha B})$, such that $\omega_1^t \notin \{\omega_1, \omega_2, \cdots, \omega_m\}$.
Thus there not exist $s \in S_m$, such that $(t, t, \cdots, t)s \in G_{\alpha B}$. A contraduction.

(2.2) Suppose $\rho(G_{\alpha B}) < \rho(G_B)$.
As $G_{\alpha B}$ is a maxical subgroup of $G_B, \langle G_{\alpha B}, g\rangle = G_B$, for all $g \in G_B \setminus G_{\alpha B}$.
The map $\rho : Aut(T) \times S_m \to Aut(T)$ is a group homomorphism.
Then $\rho(G_B) = \rho(\langle G_{\alpha B}, g\rangle) = \langle \rho(G_{\alpha B}), \rho(g)\rangle$, for all $g \in G_B \setminus G_{\alpha B}$.
If $\rho(g) \in \rho(G_{\alpha B})$, then $\rho(G_{\alpha B}) = \rho(G_B)$.
A contradiction.
Then $\rho(g) \notin \rho(G_{\alpha B})$.
Thus $\rho(G_{\alpha B})$ is a maximal subgroup of $\rho(G_B)$.

2.2.1) Suppose $\rho(G_{\alpha B}) \cap T \neq 1$.
Consider the group $H = \langle D(\rho(G_{\alpha B}) \cap T), G_{\alpha B} \rangle$.
Let $1 \neq t \in \rho(G_{\alpha B}) \cap T$. Then there exist unique $s_t \in S_m$, such that $(t, t, \cdots, t)s_t \in G_{\alpha B}$.
As $N_{\alpha B} = 1, s_t \neq 1$, for all $1 \neq t \in \rho(G_{\alpha B}) \cap T$.
Thus $s_t \in H$, but $s_t \notin G_{\alpha B}$. So, $G_{\alpha B} < H$.
As $\rho(H) = \rho(G_{\alpha B}) < \rho(G_B), H \neq G_B$.
So $G_{\alpha B} < H < G_B$. A contradiction.

2.2.2) Suppose $\rho(G_{\alpha B}) \cap T = 1$.
Then $\rho(G_B) = T : \rho(G_{\alpha B})$. And
$|T| = \frac{|\rho(G_B)|}{|\rho(G_{\alpha B})|} \leqslant \frac{|G_B|}{|\rho(G_{\alpha B})|} = \frac{|G_B|}{|G_{\alpha B}|} = \frac{|N_B|}{|N_{\alpha B}|} = k \leqslant |N_B| = |T|$.
Thus $G_B \cong \rho(G_B) \leqslant Aut(T)$, and
$G_B/N_B \cong \rho(G_B)/\rho(N_B) \leqslant Aut(T)/T$ is solvable.
Then G_B has unique minimal normal subgroup N_B and G_B is primitive and faithful on $\Gamma(B)$ of type AS.
But N_B is regular on $\Gamma(B)$. A contradiction.

Thank you!