Prime graphs and actions of primitive groups

Melissa Lee
University of Auckland

9th June 2022

Outline

Bases of Permutation Groups

Definitions

Let $G \leq \mathrm{Sym}(\Omega)$ be a permutation group.

A base for G is a subset $B \subseteq \Omega$ such that $\bigcap_{b \in B} G_b$ is trivial.

The size of the smallest base for G on Ω is called the base size of G, and is denoted $b(G)$.

Examples

- $G = S_n$ acting on $\{1, \dots, n\}$ $b(S_n) = n-1$
- G acting on itself by right multiplication $b(G) = 1$

In this talk, we will assume all groups have finite order, and that our actions are faithful.

A simple bound

Lemma

The base size $b(G)$ of the action of a permutation group G on a set Ω of size n satisfies

$$b(G) \geq \frac{\log|G|}{\log n}$$

Proof: The pointwise stabiliser of B is trivial, so the action of each element $g \in G$ on Ω is determined by its action on the base B. So for any base B,

$$|G| < n^{|B|} \quad \text{and so} \quad |G| < n^{b(G)}$$

Therefore

$$b(G) \geq \frac{\log|G|}{\log n}$$

□

Uses for bases

- Bases were formally defined by Charles Sims in the early 1970s in the context of computational group theory.

- **How do we store a permutation group of large degree efficiently?**

- Knowing a base B for a permutation group G acting on a set Ω allows us to store each element of G as a $|B|$-tuple, rather than an $|\Omega|$-tuple.

Uses for bases

Given a base $\{b_1, b_2, \ldots, b_k\}$ for $G \leq \mathrm{Sym}(\Omega)$, define

$G_i = G_{b_1, .. b_i}$

$$G_0 = G \geq G_1 \geq G_2 \cdots \geq G_k = 1$$

Let U_i be a set of (right) coset representatives of G_{i-1}/G_i. Then every $g \in G$ can be written uniquely as

$$g = u_k u_{k-1} \cdots u_1 \qquad u_i \in U_i$$

This technique also allows efficient computation of orbits, stabilisers, membership testing etc.

Arbitrary permutation groups

Theorem (Pyber, 1993)

There exists a universal constant $C > 0$ such that *almost all* subgroups $G \leq S_n$ have

$$b(G) > C \cdot n$$

- Therefore, we cannot find any *good* (i.e., non-linear) upper bounds for $b(G)$ that hold for arbitrary permutation groups G.
- In order to make meaningful statements about the size of $b(G)$, we usually restrict our focus to primitive groups.

Bases for primitive groups

1. S_n acting on $\{1, 2, \ldots, n\}$

$b(G) = n-1$

$$\frac{\log|G|}{\log|\Omega|} = \frac{\log n!}{\log n} > \frac{n}{2}$$

$$b(G) \leq 2\frac{\log|G|}{\log|\Omega|}$$

2. S_n acting on pairs of elements from $\{1, \ldots n\}$.

Halasi (2012) $b(G) = \left\lceil \frac{2n-2}{3} \right\rceil \quad n \geq 4$

$$\frac{\log|G|}{\log|\Omega|} \geq \frac{\log(n!)}{\log(n^2)} \approx \frac{n}{2}$$

The almost simple case

A group G is almost simple if there exists a non-Abelian finite simple group S such that $S \leq G \leq \mathrm{Aut}(S)$.

Suppose G is an almost simple group with socle G_0. A transitive action of G on a set Ω is standard if either:

1. G acts in a "subspace action"
2. $G_0 = \mathrm{soc}(G) = A_n$ acting on partitions or subsets of $\{1, \ldots, n\}$

An action of an almost simple group acting transitively on a set Ω is non-standard if it is not standard.

Almost simple primitive groups

Theorem (Burness et al., 2007–2011)

If G is a primitive almost simple group in a non-standard action, then $b(G) \leq 7$ with equality $\Leftrightarrow$ $G = M_{24}$ acting naturally on 24 pts.

Pyber's conjecture

From before:

$$n^{|B|} \geq |G| \Rightarrow \frac{\log |G|}{\log n} \leq b(G)$$

Conjecture (Pyber 1993)

Let G be a primitive permutation group of degree n. Then there exists an absolute constant c such that

$$b(G) \leq c \frac{\log|G|}{\log n}$$

Pyber's conjecture solved

Theorem (Duyan, Halasi & Maróti (2016))

There is a universal constant $c > 0$ such that $b(G)$ of a primitive permutation group G of degree n satisfies

$$b(G) \leq 45 \frac{\log|G|}{\log n} + C$$

Theorem (Halasi, Liebeck & Maróti (2019))

The base size $b(G)$ of a primitive permutation group G of degree n satisfies

$$b(G) \leq 2 \frac{\log|G|}{\log n} + 24$$

What about primitive groups with very small bases?

Base sizes of primitive groups

Can we classify the primitive groups $G \leq \mathrm{Sym}(\Omega)$ with *small* bases?

If $b(G) = 1$, i.e., *regular orbit*, then $\Rightarrow G \cong C_p$.

Problem: Saxl's programme

Classify all primitive groups G with $b(G) = 2$.

Known results

Diagonal & Twisted wreath type	Partial classification of groups with $b(G) = 2$ (Fawcett, 2013)
Almost simple type	Complete classification when $\mathrm{soc}(G)$ is • sporadic (Burness et al. 2010) or • alternating (James 2006; Burness et al. 2011). Partial answers for groups of Lie type.
Product type	Progress where point stabiliser is soluble (Burness & Huang, 2022+)

Affine type groups

$G = VH$, $H \leq GL(V)$

There has been a focus on **covering groups of almost simple groups**.

H $\quad H/Z(H)$ is almost simple

Complete classification when:

- sporadic (Fawcett et al., 2019),
- alternating (Fawcett et al., 2016), and

Theorem (Köhler and Pahlings, 2001; Lübeck, 2021)

Let $G = VH$ be a primitive group of affine type whose point stabiliser $H = H(q)$ is a covering group of an almost simple group of Lie type.

If $|H|$ and $|V|$ are **coprime**, then $b(G) = 2$, or (H, V) is one of a list of known examples.

More groups of Lie type

$G = VH$

What about cases where $|H|$ and $|V|$ are not coprime?

- Here the possible V are not well understood in general.
- Sometimes we only know the smallest (non-trivial) possibility for V.
- We must appeal to a technique which carefully counts elements in these groups and their eigenspaces.

A theorem

Theorem (L., 2021)

Let $G = VH$ be a primitive group of affine type whose point stabiliser H is a covering group of an almost simple group of Lie type.

1. If H and V **do not** have the same underlying field characteristic, then $b(G)$ is known.
2. if H, V have the same underlying characteristic and $\mathrm{soc}(H/Z(H)) \cong \mathrm{PSL}_n(q)$, then $b(G) = 2$, or (H, V) is one of several families of infinite examples.

Corollary

In either case, $b(G) \leq 5$.

The tables

$b(G)$	G	d	r	
2	$L_2(4)$	2	4	
	$2.L_2(4) \leq G \leq \mathbb{F}_5^\times \circ (2.L_2(4))$	2	5	
	$2.L_2(4) \leq G \leq \mathbb{F}_9^\times \circ (2.L_2(4))$	2	9	
	$c \circ (2.L_2(4))$, $c \mid r-1$, $c \neq 1,2$	2	11, 19	†
	$c \times L_2(4)$, $c \in \{5, 15\}$	2	16	
	$c \circ (2.L_2(4))$, $c \in \{6, 12, 24\}$	2	25	
	$c \circ (2.L_2(4))$, $c \in \{4, 28\}$	2	29	†
	$c \circ (2.L_2(4))$, $c \in \{\frac{r-1}{2}, r-1\}$	2	31, 41	†
	$\mathbb{F}_{61}^\times \circ (2.L_2(4))$	2	61	†
	$c \circ (2.L_2(4))$, $c \in \{12, 24, 48\}$	2	49	†
	$c \circ (2.L_2(4).2)$, $c \mid 24$	2	25	
	$c \circ (2.L_2(4).2i)$, $c \mid 24$, $c > 2$	2	25	
	$L_2(4) \leq G \leq \mathbb{F}_5^\times \times L_2(4).2$	3	5	
	$L_2(4) < G \leq \mathbb{F}_9^\times \times L_2(4)$	3	9	
	$\mathbb{F}_{11}^\times \times L_2(4)$	3	11	†
	$L_2(4)$	4	2	
	$L_2(4) \leq G \leq \mathbb{F}_3^\times \times L_2(4).2$	4	3	
	$L_2(4) < G \leq \mathbb{F}_4^\times \times L_2(4).2$	4	4	
	$\mathbb{F}_5^\times \circ (2.L_2(4))$	4	5	
	$c \times L_2(4).2$, $c \in \{2, 6\}$	4	7	†
	$c \times L_2(8)$, $c \mid 7$	2	8	
	$c \times L_2(8)$, $c = 3k \mid 63$	2	64	
	$\mathbb{F}_8^\times \times L_2(8)$	4	8	
	$\mathbb{F}_3^\times \times L_2(8)$	7	3	
	$L_2(8) \leq G \leq L_2(8).3$	8	2	
	$c \times (L_2(8).3)$, $c \in \{1, 2\}$	7	3	
	$2.L_2(9) \leq G \leq 2.L_2(9).2_2$	2	9	
	$c \circ (2.L_2(9))$, $c \in \{10, 20, 40, 80\}$	2	81	
	$c \circ (2.L_2(9)).2_2$, $c \mid 80$	2	81	
	$3.L_2(9)$	3	4	
	$L_2(9) \leq G \leq \mathbb{F}_9^\times \times L_2(9).2_2$	3	9	
	$\mathbb{F}_{16}^\times \circ (3.L_2(9))$	3	16	
	$c \circ (3.L_2(9))$, $c \in \{2, 6, 18\}$	3	19	†
	$c \circ (3.L_2(9))$, $c \in \{10, 30\}$	3	31	†
	$c \circ (3.L_2(9))$, $c \mid 24$ even	3	25	
	$c \circ (3.L_2(9).2_3)$, $c \mid 24$ even	3	25	
	$L_2(9) \leq G \leq L_2(9).2_3$	4	3	
	$L_2(9) \leq G \leq \mathbb{F}_4^\times \times L_2(9).2_1$	4	4	
	$2.L_2(9) \leq G \leq \mathbb{F}_5^\times \circ (2.L_2(9).2_1)$	4	5	
	$\mathbb{F}_7^\times \circ (2.L_2(9))$	4	7	†
	$\mathbb{F}_7^\times \circ (2.L_2(9).2_1)$	4	7	†
	$c \times L_2(9).2_1$, $c \in \{1, 7\}$	4	8	
	$L_2(9).2_1 \leq G \leq L_2(9).2^2$	4	9	
	$2 \times L_2(9) \leq G \leq \mathbb{F}_9^\times \times L_2(9).2^2$	4	9	
	$L_2(9) < G \leq \mathbb{F}_5^\times \circ (L_2(9).2_1)$	5	5	
	$c \times L_2(9).2_1$, $c \in \{2, 3, 6\}$	5	7	$\dagger\star_1$
	$c \times L_2(9).2^2$, $c \in \{1, 2\}$	9	3	‡
	$c \times L_2(11)$, $c \mid (r-1)$	5	3,4,5	
	$2.L_2(11)$	6	3	
	$L_2(11) \leq G \leq L_2(11).2$	10	2	
	$2.L_2(13)$	6	3	
	$\mathbb{F}_4^\times \times L_2(13)$	6	4	
	$c \times L_2(13)$, $c \in \{1, 2\}$	7	3	
	$L_2(13).2$	14	2	
	$L_2(17)$	8	2	
	$L_2(23)$	11	2	
	$L_2(25) \leq G \leq L_2(25).2_2$	12	2	
	$L_2(31)$	15	2	
	$c \circ (2.L_3(2))$, $c \mid 6$	2	7	
	$c \circ (2.L_3(2))$, $c \mid 48$, $c \neq \{2, 3, 6\}$	2	49	
	$c \circ (2.L_3(2).2)$, $c \mid 48$, $c \neq \{2, 3, 6\}$	2	49	
	$c \times L_3(2)$, $c \mid 3$	3	4	
	$L_3(2) \leq G \leq \mathbb{F}_7^\times \times L_3(2).2$	3	7	
	$\mathbb{F}_8^\times \times L_3(2)$	3	8	

$b(G)$	G	d	r	
2	$c \times L_3(2)$, $1 \neq c \mid 8$	3	9	
	$c \times L_3(2)$, $c \in \{2, 10\}$	3	11	†
	$L_3(2) \leq G \leq \mathbb{F}_3^\times \times L_3(2).2$	6	3	
	$L_3(2) \leq G \leq L_3(2).2$	8	2	
	$L_3(3) \leq G \leq L_3(3).2$	12	2	
	$4_2.L_3(4) \leq G \leq \mathbb{F}_9^\times \circ (4_2.L_3(4).2_2)$	4	9	
	$2.L_3(4) \leq G \leq \mathbb{F}_9^\times \circ (2.L_3(4).2^2)$	6	9	
	$6.L_3(4) \leq G \leq 6.L_3(4).2_1$	6	7	
	$4_1.L_3(4).2_3$	8	5	
	$L_4(2) \leq G < \mathbb{F}_3^\times \times L_4(2).2$	7	3	
	$L_4(2) \leq G \leq \mathbb{F}_5^\times \times L_4(2).2$	7	5	
	$L_4(2) < G \leq \mathbb{F}_7^\times \times L_4(2).2$	7	7	
	$L_4(2).2 < G \leq \mathbb{F}_9^\times \times L_4(2).2$	7	9	
	$2.L_4(2)$	8	3	
	$\mathbb{F}_5^\times \circ (2.L_4(2).2)$	8	5	
	$L_4(3) \leq G \leq L_4(3).2_2$	26	2	
	$U_3(3)$	3	9	
	$c \times U_3(3)$, $c = 2k \mid 80$	3	81	
	$U_3(3) \leq G \leq \mathbb{F}_4^\times \times U_3(3).2$	6	4	
	$U_3(3) < G \leq \mathbb{F}_5^\times \times U_3(3)$	6	5	†
	$\mathbb{F}_7^\times \times U_3(3)$	6	7	
	$c \times U_3(3).2$, $c \in \{2, 6\}$	6	7	
	$U_3(3) \leq G \leq \mathbb{F}_3^\times \times U_3(3).2$	7	3	
	$c \times U_3(3).2$, $c \mid 4$	7	5	$\dagger\star_2$
	$U_3(3) \leq G \leq \mathbb{F}_5^\times \times U_3(3)$	7	5	$\dagger *$
	$U_3(3) \leq G \leq U_3(3).2$	14	2	
	$\mathbb{F}_3^\times \times U_3(4)$	12	3	
	$U_3(5) < G \leq U_3(5).S_3$	20	2	
	$c \circ (2.U_4(2))$, $c \mid (r-1)$	4	7,13	†
	$2.U_4(2) \leq G \leq \mathbb{F}_9^\times \circ (2.U_4(2).2)$	4	9	
	$c \times U_4(2)$, $c \mid 15$	4	16	
	$c \circ (2.U_4(2))$, $c \mid 18$, $c > 2$	4	19	†
	$c \circ (2.U_4(2))$, $c = 3k \mid 24$	4	25	
	$2.U_4(2) \leq G \leq \mathbb{F}_{27}^\times \circ (2.U_4(2).2)$	4	27	
	$\mathbb{F}_r^\times \circ (2.U_4(2))$	4	31,37	†
	$12 \circ (2.U_4(2))$	4	37	†
	$c \times U_4(2)$, $c \in \{3, 9, 21, 63\}$	4	64	
	$U_4(2)$	5	7	†
	$U_4(2) < G \leq \mathbb{F}_r^\times \times U_4(2)$	5	7,13	†
	$U_4(2) \leq G \leq \mathbb{F}_9^\times \times U_4(2).2$	5	9	
	$c \times U_4(2), c \in \{6, 18\}$	5	19	†
	$c \times (U_4(2))$, $c \in \{2, 26\}$	5	27	
	$c \times (U_4(2).2)$, $c \mid 26$	5	27	
	$c \times U_4(2)$, $c \in \{1, 3\}$	6	4	
	$U_4(2) \leq G \leq \mathbb{F}_5^\times \times U_4(2).2$	6	5	
	$U_4(2) \leq G \leq \mathbb{F}_7^\times \times U_4(2).2$	6	7	†
	$U_4(2) \leq G \leq \mathbb{F}_8^\times \times U_4(2).2$	6	8	
	$\mathbb{F}_{11}^\times \times U_4(2)$	6	11	†
	$U_4(2).2 < G \leq \mathbb{F}_r^\times \times U_4(2).2$	6	11,13	$\dagger\star_3$
	$c \times (U_4(2).2)$, $c \mid 15$	6	16	
	$U_4(2) \leq G \leq \mathbb{F}_3^\times \times U_4(2).2$	10	3	
	$U_4(2) \leq G \leq U_4(2).2$	14	2	
	$6_1.U_4(3) \leq G \leq 6_1.U_4(3).2_2$	6	7	
	$c \circ (3_1.U_4(3))$, $c \mid 15$	6	16	
	$c \circ (3_1.U_4(3).2_2)$, $c \mid 15$	6	16	
	$c \circ (6_1.U_4(3))$, $c \mid (r-1)$	6	13,19	†
	$\mathbb{F}_{31}^\times \circ (6_1.U_4(3))$	6	31	†
	$c \circ (6_1.U_4(3).2_2)$, $c \mid (r-1)$	6	13,19	†
	$c \circ (6_1.U_4(3).2_2)$, $c \mid (r-1)$	6	31,37	†
	$c \circ (6_1.U_4(3))$, $c \mid 24$	6	25	
	$c \circ (6_1.U_4(3).2_2)$, $c \mid (r-1)$	6	25,49	
	$U_4(3) \leq G \leq U_4(3).D_8$	20	2	
	$U_5(2) \leq G < \mathbb{F}_3^\times \times U_5(2).2$	10	3	
	$U_5(2) \leq G \leq \mathbb{F}_5^\times \times U_5(2).2$	10	5	

TABLE 1.2. Pairs (G, V) in Theorem 1.1 with $b(G) > 1$, Part I.

IBIS groups

IBIS groups

An ordered base $(b_1, \ldots b_t)$ for a finite group G is **irredundant** if

$$G > G_{b_1} > G_{b_1, b_2} > \ldots > G_{b_1, \ldots b_t} = 1$$

We say that G is a Irredundant Bases of Invariant Size (IBIS) group if all of its irredundant bases are of the same size.

Theorem (Cameron & Fon-Der-Flaass, 1995)

The following are equivalent.

1. G is IBIS
2. All irredundant bases for G are preserved by reordering
3. All irredundant bases for G are bases of a matroid

Some problems

Problem

Classify all IBIS groups

(Easier?) Problem

Classify all primitive IBIS groups!

Primitive IBIS groups

Theorem (Lucchini, Morigi, Moscatiello, 2021+)

If G is a primitive IBIS group, then one of the following holds.

1. G is of diagonal type.
2. G is almost simple.
3. G is an affine type group.

Theorem (Lucchini, Morigi, Moscatiello, 2021+)

G is a primitive IBIS group of diagonal type if and only if, for some $f \geq 2$,

$$G \cong PSL_2(2^f) \times PSL_2(2^f)$$

Almost simple IBIS groups

Lemma (Lucchini, Morigi, Moscatiello, 2021+)

There are no primitive IBIS groups with non-Abelian socle and base size 2.

Theorem (Burness et al.)

The almost simple primitive groups with sporadic or alternating socle and $b(G) > 2$ are known.

Proposition (L. 2021+, Spiga 2022+)

The almost simple primitive IBIS groups G with sporadic or alternating socle and point stabiliser H are known.

Almost simple IBIS groups

Problems

- Finish classification of almost simple IBIS grps
- Consider affine type IBIS grps

Prime graphs

Prime graphs

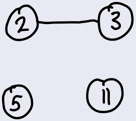
The **prime graph** or **Gruenberg-Kegel graph** $\Gamma(G)$ of a finite permutation group $G \leq \mathrm{Sym}(\Omega)$ has:

- Vertices: the primes dividing $|G|$,
- Edges: $p \sim r$ if there is an element of order pr in G.

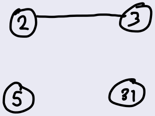
Examples

(a) p-group

(b) M_{11}



(c) $\mathrm{PSL}_2(31)$



Recognisability of prime graphs

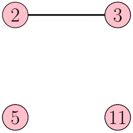
We say that a finite group G is:

- **Recognisable** by prime graph if $\Gamma(G) \cong \Gamma(H) \Leftrightarrow G \cong H$.
- k-**recognisable** by prime graph (for $k \in \mathbb{N}$) if there are exactly k groups H such that $\Gamma(G) \cong \Gamma(H)$.
- **Unrecognisable** by prime graph if there are infinitely many groups H such that $\Gamma(G) \cong \Gamma(H)$.

(a) p-group

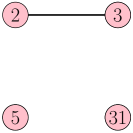
unrecog.

$PSL_2(11)$



(b) M_{11}

2-recog.



(c) $\mathrm{PSL}_2(31)$

recog.

Recognisability of prime graphs

Theorem (Cameron & Maslova, 2021+)

If G is a group that is k-recognisable by prime graph, then G is almost simple with non-abelian socle.

Problem

For each almost simple group G, determine whether G is recognisable by prime graph.

Progress - Sporadic groups

Author(s)	Recognisable	k-recognisable	Unrecognisable
Hagie (2003)	J_1, M_{22}, M_{23}, M_{24}, Co_2	M_{11} $k=2$	M_{12}, J_2
Zavarnitsine (2006)	J_4		
Kondrat'ev (2019, 2020)	Ru, J_3, Suz, $O'N$, Ly, Th, Fi_{23}, Fi_{24}'	HN HS $k=2$	McL, Co_3
L. & Popiel (2021+)	Co_1, B, M		

Progress - Lie type groups

Many partial results, almost all for simple groups.

Author(s)	Recognisable	k-recognisable
Khosravi et al. (2007)	$PSL_2(p)$ $p>11$ $p \not\equiv 1\ (12)$	
Khosravi (2008)	$PSL_2(p^k)$ p odd $k>1$	
Zavarnitsine (2006)	${}^2G_2(q)$	
Ahanjideh et al. (2013)		$P\Omega_{2m+1}(3)$ $PSp_{2k}(3)$ $k=2$

Progress - alternating and symmetric groups

Also many partial results by Gorshkov et al., Khosravi et al., Staroletov and others ...

Theorem (Gorshkov et al., 2019)

If G is a group with the same prime graph as either A_n or S_n with $n \geq 19$, then there exists $K \trianglelefteq G$ and $t \in \mathbb{N}$ such that

$$A_t \leq G/K \leq S_t,$$

and K is divisible by at most one prime greater than $n/2$.

Prop.

~~Theorem~~ (L., 2021+)

1. The symmetric group S_n, $n \geq 5$ is unrecognisable by prime graph.
2. If A_n, $n \geq 5$ is recognisable by prime graph, then either
    - n, $n-2$ are prime
    - $n-1$, $n-3$ are prime

Some other open problems

- What graphs can be realised as the prime graphs of some finite group?
  - labelled
  - unlabelled
- The maximum number of almost simple groups with the same prime graph?

Extremely primitive groups

Extremely primitive groups

A finite, non-regular primitive permutation group $G \leq \mathrm{Sym}(\Omega)$ is **extremely primitive** if a point stabiliser G_α acts primitively on each of its orbits on $\Omega \setminus \alpha$.

Theorem (Mann, Praeger & Seress, 2007)

An extremely primitive group is either almost simple, or of affine type.

Theorem (Burness, Praeger, Seress, 2012; Burness & Thomas, 2021)

An extremely primitive **almost simple** group $G \leq \mathrm{Sym}(\Omega)$ belongs to one of five infinite families, or is one of 21 further examples.

The almost simple extremely primitive groups

G_0	$H \cap G_0$	Rank	Conditions
Alt_n	$(\mathrm{Sym}_{n/2} \wr \mathrm{Sym}_2) \cap G_0$	$\frac{1}{4}(n+2)$	$n \equiv 2 \pmod 4$
Alt_n	Alt_{n-1}	2	$G = \mathrm{Sym}_n$ or Alt_n
Alt_6	$\mathrm{L}_2(5)$	2	$G = \mathrm{Sym}_6$ or Alt_6
Alt_5	D_{10}	2	
$\mathrm{L}_2(q)$	P_1	2	
$\mathrm{L}_2(q)$	$D_{2(q+1)}$	$\frac{1}{2}q$	$G = G_0$, $q+1$ is a Fermat prime
$\mathrm{Sp}_n(2)$	$\mathrm{O}_n^{\pm}(2)$	2	$n \geqslant 6$
$\mathrm{U}_4(3)$	$\mathrm{L}_3(4)$	3	See Remark 1(v)
$\mathrm{L}_3(4)$	Alt_6	3	See Remark 1(vi)
$\mathrm{L}_2(11)$	Alt_5	2	$G = G_0$
$G_2(4)$	J_2	3	
M_{11}	Sym_6	2	
M_{11}	$\mathrm{L}_2(11)$	2	
M_{12}	M_{11}	2	$G = G_0$
M_{22}	$\mathrm{L}_3(4)$	2	
M_{23}	M_{22}	2	
M_{24}	M_{23}	2	
J_2	$\mathrm{U}_3(3)$	3	
HS	M_{22}	3	
HS	$\mathrm{U}_3(5)$	2	$G = G_0$
Suz	$G_2(4)$	3	
McL	$\mathrm{U}_4(3)$	3	
Ru	${}^2F_4(2)$	3	
Co_2	$\mathrm{U}_6(2)$	3	
Co_2	McL	6	
Co_3	McL	2	

TABLE 1. The extremely primitive almost simple groups

Affine extremely primitive groups

Theorem (Mann, Praeger & Seress, 2007)

Let $G = VH$ be an extremely primitive affine group, where $V = V_d(p)$. Then either:

1. (H, d, p) is one of a known list of examples, or
2. $p = 2$ and H is almost simple.

Moreover, if we assume **Wall's conjecture**, then (2) reduces to a list $\mathcal{S}$ of three infinite families of (H, d, r) and 16 further possibilities.

Theorem (Burness & Thomas, 2020+)

No example in $\mathcal{S}$ is extremely primitive!

Observation: If $G = VH$ with H almost simple and H has a regular orbit on V, then G is not extremely primitive.

Theorem (Burness & L., 2021+)

There are no further extremely primitive groups.

Thank you!