Bases for primitive permutation groups

Hong Yi Huang

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University of BRISTOL

Outline

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- $G = \mathrm{GL}(V)$, $\Omega = V$ and Δ contains a basis of V.

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Note. There exists a base of size $m \iff G$ has a regular orbit on Ω^m.

Base sizes

Observation. If Δ is a base and $x, y \in G$, then

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Question. How small can a base be?

- A small base Δ provides an efficient way to store the elements of G, using $|\Delta|$-tuples rather than $|\Omega|$-tuples.

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Other applications:

- Minimal dimension
- 2-generation of finite groups
- Extremely primitive groups
- Some graphs defined on groups

First bounds

Let $\Delta = \{\alpha_1, \ldots, \alpha_{b(G)}\}$ be a base and set $G^{(k)} = \bigcap_{i=1}^{k} G_{\alpha_i}$. Then

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Note. The former example is primitive, while the latter is imprimitive.

Pyber's conjecture

Let G be primitive with degree n.

Conjecture (Pyber, 1993)

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Duyan, Halasi & Maróti, 2018: Pyber's conjecture is true.

Halasi, Liebeck & Maróti, 2019: $b(G) \leqslant 2 \log_n |G| + 24$.

Other bounds in the primitive setting

Soluble groups:

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G is called standard if either

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Burness et al., 2007-11: $b(G) \leqslant 7$ if G is non-standard.

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- G is primitive iff n is a prime.

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- G non-standard with T Lie type: Partial results

  e.g. T classical, $G_\alpha \in \mathcal{S}$ **(Burness, Guralnick & Saxl, 2014)**

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- The case $G < L \wr P$: Just initiated **(Burness & H, 2022+)**

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Outline

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Definition (Burness & Giudici, 2020)

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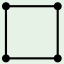
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 

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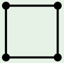
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
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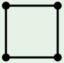
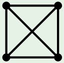
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
- $G = C_4 = \langle (1234) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 

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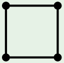
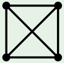
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
- $G = C_4 = \langle (1234) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
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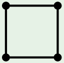
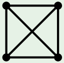
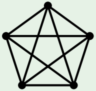
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
- $G = C_4 = \langle (1234) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
- $G = D_{10} = \langle (12345), (25)(34) \rangle$, $\Omega = \{1, 2, 3, 4, 5\}$: $\Sigma(G) =$ 

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Let $G = \mathrm{PGL}_2(q)$ and Ω be the set of distinct pairs of 1-spaces in $\mathbb{F}_q^2$.

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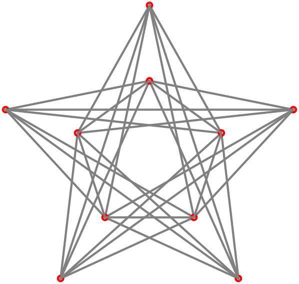
For example, when $q = 4$ we have the complement of the Petersen graph.



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- $G \in \mathcal{B}$ is simple $\implies \omega(G) \geqslant 5$ or $(G, G_\alpha) = (A_5, S_3)$;

Other invariants

$\mathcal{S} := \{$almost simple primitive groups with soluble stabilisers$\}$

- **Li & Zhang, 2011:** $\mathcal{S}$ is completely known ✓

$\mathcal{B} := \{G \in \mathcal{S} \mid b(G) = 2\}$

- **Burness, 2021:** $\mathcal{B}$ is completely known ✓
- **Burness & H, 2021+:** The common neighbour property for $\mathcal{B}$ ✓

Clique number: maximal size $\omega(G)$ of a complete subgraph of $\Sigma(G)$.

Independence number: clique number $\alpha(G)$ of the complement of $\Sigma(G)$.

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Probabilistic methods

Recall that $v(G) = r(G)|G_\alpha|$ and

$$Q(G) = 1 - \frac{r(G)|G_\alpha|}{n} = 1 - \frac{v(G)}{n}$$

is the probability that a random pair in Ω is not a base for G.

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Example

If $G = \mathrm{PGL}_2(q)$ and $G_\alpha = D_{2(q-1)}$, then $Q(G) \to 1$ as $q \to \infty$. But $\Sigma(G) = J(q+1, 2)$ still has the common neighbour property.

A strong conjecture

Let $\Sigma(\alpha)$ be the set of neighbours of α in $\Sigma(G)$.

Conjecture (Burness & H, 2022+)

G primitive and $\alpha, \beta \in \Omega \implies \Sigma(\alpha)$ meets every regular G_β-orbit.

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Outline

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- More results on blow-ups? (e.g. product type)

Thank you!