Reflexible Orientably-regular Cayley Maps on Nonabelian Simple Groups

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1. Introduction: Orientably-regular Cayley maps
2. Some useful results
   - Skew-morphisms of nonabelian simple groups
   - Classification of bi-quasiprimitive d-groups
3. Main theorem: Characterization of Aut($\mathcal{M}$)
4. Method to tell reflexible ones

Outline

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Maps

Here by a *surface* we mean a connected, closed 2-manifold without boundary.

Definition

An embedding of a graph $\Gamma = (V, E)$ into a surface is called a **2-cell embedding** if E divides the surface into discs, called **faces**.
We denote the set of faces by F, and the triple (V, E, F) is a (2-cell) **map**.

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In other words, "drawing" a graph $\Gamma = (V, E)$ into a surface S such that

1. any two edges do not intersect except for the end points,
2. E divides the surface S into discs.

Automorphism groups of maps

Definition

An **automorphism** of a map $\mathcal{M} = (V, E, F)$ is a permutation of the flags of $\mathcal{M}$ which preserves the incidence relation.
An **orientation-preserving(reversing) automorphism** is a map automorphism preserving(reversing) the orientation given by the underlying surface.

All (orientation-preserving) automorphisms of $\mathcal{M}$ form group $\mathrm{Aut}(\mathcal{M})$ (resp. $\mathrm{Aut}^+(\mathcal{M})$). Clearly, $\mathrm{Aut}^+(\mathcal{M}) \leqslant \mathrm{Aut}(\mathcal{M}) \leqslant \mathrm{Aut}(\Gamma)$.

Definition

An orientable map $\mathcal{M}$ is called **reflexible** if there exists an orientation-reversing automorphism of $\mathcal{M}$.

Lemma

The action of $\mathrm{Aut}(\mathcal{M})$ *(resp.* $\mathrm{Aut}^+(\mathcal{M})$*) on flags (resp. arcs) are semi-regular.*

Definition

$\mathcal{M}$ is called **(orientably-)regular** if $\mathrm{Aut}(\mathcal{M})$ (resp. $\mathrm{Aut}^+(\mathcal{M})$) is transitive on the set of flags (resp. arcs).

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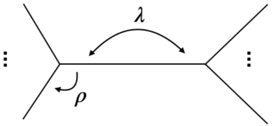
An orientably-regular map can be identified with a group presentation

$$\mathrm{Aut}^+(\mathcal{M}) = \langle \lambda, \rho \mid \lambda^2, \rho^l, (\lambda\rho)^m, \cdots \rangle \quad (1)$$

where its edges, vertices and faces represented by the right cosets of the subgroups $\langle\lambda\rangle$, $\langle\rho\rangle$ and $\langle\lambda\rho\rangle$ respectively.



Similarly, for a regular map,

$$\mathrm{Aut}(\mathcal{M}) = \langle x, y, z \mid x^2, y^2, z^2, (xy)^2, (yz)^l, (xz)^m, \cdots \rangle \tag{2}$$

Lemma

1. *A regular map $\mathcal{M}$ with $\mathrm{Aut}(\mathcal{M}) = \langle x, y, z \rangle$ is orientable if and only if the subgroup $\langle xy, yz \rangle$ is of index 2 in $\mathrm{Aut}(\mathcal{M})$. In this case $\mathcal{M}$ is orientably-regular and $\mathrm{Aut}^+(\mathcal{M}) = \langle \lambda, \rho \rangle$ where $\lambda = xy$, $\rho = yz$.*
2. *An orientably-regular map $\mathcal{M}$ with $\mathrm{Aut}^+(\mathcal{M}) = \langle \rho, \lambda \rangle$ is regular if and only if $\mathcal{M}$ is reflexible, equivalently, there exists an automorphism of $\mathrm{Aut}^+(\mathcal{M})$ that fixes λ and inverts ρ.*

Cayley maps

Definition

A **Cayley map** $\mathrm{CM}(G, S, \rho)$ is a map which has a Cayley underlying graph $Cay(G, S)$ and the same orientation at each vertex is given by a cyclic permutation ρ on S.
Further, it is called **balanced** if $\rho(s^{-1}) = (\rho(s))^{-1}$, $\forall s \in S$.

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Theorem (Richter, Širáň et al., 2005 [5])

An orientable map $\mathcal{M}$ is a Cayley map $\mathrm{CM}(G, S, \rho)$ if and only if G is a subgroup of $\mathrm{Aut}^{+}(\mathcal{M})$ acting regularly on the set of vertex.

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Theorem (Škoviera and Širáň, 1992 [6])

An orientably-regular Cayley map $\mathcal{M} = \mathrm{CM}(G, S, \rho)$ is balanced if and only if $G \trianglelefteq \mathrm{Aut}^+(\mathcal{M})$.

Orientably-regular Cayley maps and Skew-morphisms

Let G be a finite group.

Definition

A **skew-morphism** of G is a permutation φ of G satisfying $1_G^\varphi = 1_G$ and there exists a function $\pi : G \to \mathbb{Z}_{|\varphi|}$ such that $(gh)^\varphi = g^\varphi h^{\varphi^{\pi(g)}}$ for all $h \in G$.

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Theorem (Jajcay and Širáň, 2002 [3])

A Cayley map $\mathrm{CM}(G, S, \rho)$ is orientably-regular if and only if G admits a skew-morphism φ such that $\varphi|_S = \rho$.

Theorem (Jajcay and Širáň, 2002 [3])

A group G admits an orientably-regular Cayley map $CM(G, S, \rho)$ if and only if there exists a skew-morphism φ of G that has a symmetric orbit S that generates the group G.

Orientably-regular Cayley maps on the following groups are classified:

Cyclic groups[2]
(Marston Conder and Thomas Tucker, 2014)
reflexible ⇔ anti-balanced

Dihedral groups[4]
(István Kovács and Young Soo Kwon, 2021)
a list of reflexible ones is given

Nonabelian characteristically simple groups[1]
(Jiyong Chen, Shaofei Du and Cai Heng Li, 2022)
reflexibility is not determined yet

Problem

Determine the reflexibility of orientably-regular Cayley maps on nonabelian (characteristically) simple groups.

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Problem

Classify the reflexible orientably-regular Cayley maps on nonabelian simple groups (according to $\mathrm{Aut}(\mathcal{M})$*).*

Outline

Skew-morphisms of nonabelian simple groups

Suppose G is a non-abelian simple group.

Theorem (Chen, Du and Li, 2022 [1])

Suppose σ is a skew-morphism of G. And $X := G\langle\sigma\rangle$. Then one of the following holds:

1. $X = G : \langle\sigma\rangle$ *and* $\sigma \in \mathrm{Aut}(G)$; *or*
2. $(X, G) = (\mathrm{PSL}(2, 11), A_5)$, (M_{23}, M_{22}), *or* (A_{m+1}, A_m) *with* $m \geqslant 6$ *even.*

Remark: In the second case above, $X \curvearrowright \Omega = [X : G]$ primitively.

Corollary (Chen, Du and Li, 2022 [1])

Suppose $\mathcal{M}$ is an orientably-regular Cayley map on G. Then either $\mathcal{M}$ is a balanced Cayley map on G, or $\mathrm{Aut}^+(\mathcal{M})$ is simple and $G = M_{22}$, A_5, A_m with $m \geqslant 6$ even.

Classification of bi-quasiprimitive d-groups

Definition

A permutation group X is called **bi-quasiprimitive** if all its minimal normal subgroups has exactly two orbits.

Definition

If Ω has a non-trivial X-invariant partition $\Omega = U \cup W$ such that $X_U = X_W$ is primitive on U and W, then X is called **bi-primitive**.

Definition

A permutation group X is called a **d-group** if it has a dihedral regular subgroup.

Theorem (Song, Li and Zhang, 2014 [7])

Suppose $X \leqslant \mathrm{Sym}(\Omega)$ is a bi-quasiprimitive d-group which has a regular subgroup isomorphic to D_{2n}. Then either the action of X on Ω induces an orbital graph $\mathbf{K}_{n,n}$, or X is bi-primitive and divided into two cases:

1. *$n = p$, $G = D_{2p}$, $X = \mathbb{Z}_p : \mathbb{Z}_{2k} \leqslant \mathrm{AGL}(1, p)$, and $X_\omega = \mathbb{Z}_k$ with k odd;*
2. *(X, G, X_ω) is as followings:*

X	G	X_ω	Condition
D_4	D_4	1	
$\mathrm{S}_{2\ell+1}$	$\mathrm{D}_{4\ell+2}$	$\mathrm{A}_{2\ell}$	ℓ is odd
$\mathrm{S}_{4\ell}$	$\mathrm{D}_{8\ell}$	$\mathrm{A}_{4\ell-1}$	
$\mathrm{PGL}(2, r^f).e$	$\mathrm{D}_{2(r^f+1)}$	$\mathbb{Z}_r^f{:}\mathbb{Z}_{\frac{r^f-1}{2}}.\mathbb{Z}_e$	$e \mid f$, $r^f \equiv 3 \pmod 4$
$\mathrm{PGL}(d, r^f).e.2$	$\mathrm{D}_{2\frac{r^{df}-1}{r^f-1}}$	P_1	$d \geqslant 3$, $X \nleqslant \mathrm{P\Gamma L}(d, q)$
$\mathrm{PGL}(2, 11)$	D_{22}	A_5	
$\mathrm{M}_{12}.2$	D_{24}	M_{11}	

Remark: The following fact is part of a lemma of the above classification theorem.

Lemma

Suppose X is a bi-quasiprimitive d-group and $M \trianglelefteq_{min} X$ with orbits Δ, Δ'. If $X^+ := X_\Delta = X_{\Delta'}$ acts faithful both on Δ and Δ', then X is bi-primitive.

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Main theorem

Theorem

Let $\mathcal{M}$ be a reflexible orientably-regular Cayley map on a non-abelian simple group G with full automorphism group $\mathrm{Aut}(\mathcal{M})$.
Then one of the followings holds.

1. $\mathrm{Aut}(\mathcal{M}) \cong G : D_{2n}$, *i.e.,* $\mathcal{M}$ *is balanced;*
2. $\mathrm{Aut}(\mathcal{M})$ *is a bi-primitive d-group, actually,* $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{PGL}(2, 11)$ *or* S_l, *with* $G = A_5$ *or* A_{l-1} *where* $l = 4t + 3$, $t \in \mathbb{N}_+$;
3. $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}^+(\mathcal{M}) \times \mathbb{Z}_2$. *In this case,* $\mathrm{Aut}^+(\mathcal{M}) \cong A_s$ *where* $s = 4t + 5$, $t \in \mathbb{N}_+$.

Corollary

Reflexible orientably-regular Cayley maps on M_{22} *are balanced.*

Proof of the main theorem: balanced

Recall that $\mathrm{Aut}(\mathcal{M}) = \mathrm{Aut}^+(\mathcal{M}) : \langle \tau \rangle$.
Since G is simple, G is either normal or core-free in $\mathrm{Aut}^+(\mathcal{M})$.

1. $G \trianglelefteq \mathrm{Aut}^+(\mathcal{M}) \Longrightarrow \mathcal{M}$ is balanced and $\mathrm{Aut}(\mathcal{M}) = (G : \langle \sigma \rangle) : \langle \tau \rangle$.
2. G is core-free in $\mathrm{Aut}^+(\mathcal{M}) \Longrightarrow G = M_{22}$, A_5 or A_{2m} with $m \geqslant 3$.

We focus on the second case and search for $\mathrm{Aut}(\mathcal{M})$.

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We focus on the second case and search for $\mathrm{Aut}(\mathcal{M})$.

The following actions are faithful.

1. $\mathrm{Aut}(\mathcal{M}) \curvearrowright \Omega := [\mathrm{Aut}(\mathcal{M}) : G]$
2. $\mathrm{Aut}^+(\mathcal{M}) \curvearrowright \Delta := [\mathrm{Aut}^+(\mathcal{M}) : G]$
3. $\mathrm{Aut}^+(\mathcal{M}) \curvearrowright \Delta' := \{R\tau \mid R \in \Delta\} = \Omega \backslash \Delta$

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3. $\mathrm{Aut}^+(\mathcal{M}) \curvearrowright \Delta' := \{R\tau \mid R \in \Delta\} = \Omega \backslash \Delta$

By Frattini argument, $\mathrm{Aut}(\mathcal{M}) = G\,\mathrm{Aut}_v(\mathcal{M})$ for some vertex v of $\mathcal{M}$.
Then $\mathrm{Aut}_v(\mathcal{M})$ is dihedral and a regular subgroup of $\mathrm{Aut}(\mathcal{M})$,
i.e., $\mathrm{Aut}(\mathcal{M})$ is a **d-group**.

Question: $\mathrm{Aut}(\mathcal{M}) \curvearrowright \Omega := [\mathrm{Aut}(\mathcal{M}) : G]$ bi-quasiprimitive?

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Consider $N \trianglelefteq_{min} \mathrm{Aut}(\mathcal{M})$.
Then either $N \trianglelefteq \mathrm{Aut}^+(\mathcal{M})$ or $N \cap \mathrm{Aut}^+(\mathcal{M}) = 1$.
Claim that

1. $N \trianglelefteq \mathrm{Aut}^+(\mathcal{M}), \forall N \trianglelefteq_{min} \mathrm{Aut}(\mathcal{M})$.
   $\Longrightarrow \mathrm{Aut}(\mathcal{M})$ is bi-primitive.
2. $\exists N \trianglelefteq_{min} \mathrm{Aut}(\mathcal{M})$ s.t. $N \cap \mathrm{Aut}^+(\mathcal{M}) = 1$
   $\Longrightarrow \mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}^+(\mathcal{M}) \times \mathbb{Z}_2$

Proof of the main theorem: bi-primitive

Suppose $N \trianglelefteq \mathrm{Aut}^+(\mathcal{M})$, $\forall N \trianglelefteq_{min} \mathrm{Aut}(\mathcal{M})$.

Recall that $\mathrm{Aut}^+(\mathcal{M}) \curvearrowright \Delta$ primitively, hence N is transitive on Δ.
Notice that $R\tau N = RN\tau$, N is also transitive on Δ'.
$\Longrightarrow$ N has two orbits Δ and Δ'
$\Longrightarrow$ $\mathrm{Aut}(\mathcal{M}) \curvearrowright \Omega$ bi-quasiprimitive

Further, $\mathrm{Aut}(\mathcal{M})$ is bi-primitive by lemma and the fact that
$\mathrm{Aut}^+(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{M})_\Delta = \mathrm{Aut}(\mathcal{M})_{\Delta'}$ acts faithfully on both Δ and Δ'.

Thus we can read from the list that
$(\mathrm{Aut}(\mathcal{M}), G) \cong (\mathrm{PGL}(2, 11), A_5)$ or (S_l, A_{l-1}), where $l = 4t + 3$, $t \in \mathbb{N}_+$.

Proof of the main theorem: $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}^+(\mathcal{M}) \times \mathbb{Z}_2$

Suppose $\exists N \trianglelefteq_{min} \mathrm{Aut}(\mathcal{M})$ s.t. $N \cap \mathrm{Aut}^+(\mathcal{M}) = 1$.

Then $\mathrm{Aut}(\mathcal{M}) = \mathrm{Aut}^+(\mathcal{M}) \times N$ where $N = \langle r \rangle \cong \mathbb{Z}_2$.

Suppose $\tau_v \in \mathrm{Aut}(\mathcal{M})_v$ stabilizes a given vertex v and satisfying $\tau_v \rho_v = \rho_v^{-1} \tau_v$, we may consider $\tau_v = (a, r) \in \mathrm{Aut}^+(\mathcal{M}) \times \langle r \rangle$, and $\rho_v = (\rho, 1)$. Then $\tau_v \rho_v = \rho_v^{-1} \tau_v$ implies $a\rho = \rho^{-1} a$ for some $a \in \mathrm{Aut}^+(\mathcal{M})$.

This can only happen in $\mathrm{Aut}^+(\mathcal{M}) \cong A_s$ where $s = 4t + 5$, $t \in \mathbb{N}_+$.
In fact, by MAGMA, $\forall \rho \in M_{23}$ of order 23, ρ is not conjugate to ρ^{-1}.

Q.E.D.

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Method to tell reflexible ones

$\mathrm{Aut}^+(\mathcal{M}) = \langle \lambda, \rho \rangle \cong \mathrm{PSL}(2, 11)$ or A_m with $m \geqslant 7$ even.
We may assume $\rho = (1\ 2\ 3\ \cdots\ 11)$ or $(1\ 2\ 3\ \cdots\ m)$.

Step 1. List τ in $\mathrm{PGL}(2, 11)$ or S_m such that $\rho^\tau = \rho^{-1}$.
Step 2. List involutions $\lambda \in \mathrm{Aut}^+(\mathcal{M})$ s.t. $\mathrm{Aut}^+(\mathcal{M}) = \langle \lambda, \rho \rangle$.
Step 3. For each λ, check if $\exists \tau$ s.t. $\lambda^\tau = \lambda$.

Example

Simple orientably-regular Cayley maps on A_5 are reflexible.

```
X:=PGL(2,11);G:=Socle(X);r:=Classes(G)[7][3];   set Aut^+(M) and ρ
T:={t:t in X| Order(t) eq 2 and r^t eq r^(-1)};   List τ
L:={l:l in G| Order(l) eq 2 and Order(sub<G|l,r>) eq Order(G)};   List λ
i:=0;
for l in L do     check λ^τ = λ
   flag:=false;
   for t in T do
      if l*t eq t*l then flag:=true; i:=i+1; break; end if;
   end for;
   if flag eq false then l; end if;
end for;
i;#L;
```

Output: 55, 55

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