On the order of vertex-primitive 2-arc-transitive digraph besides direct cycles

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A digraph Γ is a pair $(V, \to)$ with a set V (of vertices) and an antisymmetric irreflexive binary relation $\to$ on V. (Note that by definition, if $u \to v$, then $v \not\to u$.)

- For a nonnegative integer s, an s-arc of Γ is a sequence $(v_0, v_1, ..., v_s)$ of vertices with $v_i \to v_{i+1}$ for each $i = 0, ..., s-1$. ($v_i \neq v_{i+2}$ by the definition.)
- Γ is said to be s-arc-transitive if its automorphism group $\mathrm{Aut}(\Gamma)$ acts transitively on the set of s-arcs.
- s-arc-transitive $\Rightarrow$ $(s-1)$-arc-transitive.

s-arc-transitive digraph

Let Γ be a (G, s)-arc-transitive digraph, where $s \geq 1$. Then Γ is vertex-transitive. The in-neighbours $\Gamma(v)^{-} = \{u \in V(\Gamma) | u \to v\}$ and out-neighbours $\Gamma(v)^{-} = \{w \in V(\Gamma) | v \to w\}$ has same order, called the *valency* of Γ.

- If the valency is 1, then Γ is a directed cycle, and s can be as large as possible.
- If the valency is 2 and G acts primitive on the vertex set $V(\Gamma)$ of Γ, then Γ is a (undirected) cycle of prime order. (So vertex-primitive 2-arc-transitive digraphs besides direct cycles have valency at least 3.)
- Praeger [1] constructed a family of connected s-arc-transitive digraph $C_r(v, s)$ of order rv^s and valency v, where $r \geq 3, v \geq 2, s \geq 1$. ($s$ can be as large as possible)

[1] Praeger, C. E. (1989) Highly Arc Transitive Digraphs. European Journal of Combinatorics, 10, 281-292.

Vertex-primitive s-arc-transitive digraph

(1) Vertex-primitive s-arc-transitive digraph was first studied by Praeger [1] in 1989. She studied vertex-primitive s-arc-transitive digraph in each O'Nan-Scott type. It was proved that a (G, s)-arc-transitive digraph is a directed cycle if G contains a regular normal subgroup, where $s \geq 2$. (So if G is vertex-primitive of type HA, HS, HC, TW, then the digraph is directed cycle.)

(2) Giudici and Xia studied the vertex-quasiprimitive (G, s)-arc-transitive digraph such that G is of type AS, CD, SD or PA).

  (a) They characterized those digraphs from types SD and proved those digraphs are not 3-arc-transitive. (In particular, the order of graph is $|T|^{|T|-1}$.)
  (b) They proved digraphs of type CD are direct product of digraphs of type SD and those digraphs are not 3-arc-transitive.
  (c) They proved digraphs of type PA are direct product of digraphs of type AS and those digraphs is s-arc-transitive.
  (d) They prompted a quastion:

Question: Is there an upper bound on s for vertex-primitive s-arc-transitive digraphs that are not directed cycles?

[1] Praeger, C. E. (1989) Highly Arc Transitive Digraphs. European Journal of Combinatorics, 10, 281-292.

[2] Giudici, M. & B. Xia (2017) Vertex-quasiprimitive 2-arc-transitive digraphs. Ars Mathematica Contemporanea, 14, 67-82.

(3) Giudici,Li and Xia [3] studied the vertex-primitive (G, s)-arc-transitive digraph where G is almost simple with socle $\mathrm{PSL}_n(q)$. Then proved $s \leq 2$, and provided many techniques.

(4) Pan, Wu and Yin studied the case G is almost simple with socle A_n. They proved $s \leq 2$ if the stabilizer is not a wreath product group. (The unsolved case was solved by Chen, Li and Xia[5] recently.)

(5) Chen, Giudici and Praeger [6] studied the case G is almost simple with socle ${}^2B_2(q)$ or ${}^2G_2(q)$. They proved that $s \leq 1$.

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[3] M. Giudici, C. H. Li & B. Xia (2017) An infinite family of vertex-primitive 2-arc-transitive digraphs. Journal of Combinatorial Theory, Series B, 127, 1-13.

[4] J. Pan, C. Wu & F. Yin (2020) Vertex-primitive s-arc-transitive digraphs of alternating and symmetric groups. Journal of Algebra, 544, 75-91.

[5] J. Chen, J. Li & B. Xia (2021) Bounding s for vertex-primitive 2-arc-transitive digraph of alternating and symmetric groups. arXiv:2111.06579v1

[6] L. Chen, M. Giudici, &C.E. Praeger (2021) Vertex-primitive s-arc-transitive digraphs admitting a Suzuki or Ree group. arXiv:2109.10508v1

(6) Giudici, Li and Xia [7] constructed an infinite family of vertex-primitive 2-arc-transitive digraphs that are not directed cycles (the first examples).

The graphs are $(\mathrm{PSL}_3(p^2), 2)$-arc-transitive with stabilizer A_6, where $p \equiv \pm 2 (\mathrm{mod}\ 5)$ and $p \geq 7$.

So the example is of smallest order is of order $|\mathrm{PSL}_3(7^2)|/|\mathrm{A}_6| = 30758154560$.

Question: Whether exists vertex-primitive 2-arc-transitive digraph besides directed cycles with order less than 30758154560?

(This question was introduced by Xia in the summer of 2019 when visiting our university.)

[7] M. Giudici, C.H. Li & B. Xia (2017) An infinite family of vertex-primitive 2-arc-transitive digraphs. Journal of Combinatorial Theory, Series B, 127, 1-13.

Properties of 2-arc-transitive digraphs

Proposition 1 ([GLX2019,Lemma 2.13])

For any vertex-primitive arc-transitive digraph Γ, either Γ is a directed cycle of prime length or Γ has valency at least 3.

Proposition 2 ([GLX2019,Lemma 2.14])

Let Γ be a connected G-arc-transitive digraph with arc (v, w). Let $g \in G$ such that $v^g = w$. Then each nontrivial normal subgroup of G_v is not normalized by g.

Proposition 3 ([GX2017,Lemma 2.2])

Let Γ be a G-arc-transitive digraph with a 2-arc (u, v, w). Then Γ is $(G, 2)$-arc-transitive if and only if $G_v = G_{uv}G_{vw}$. (homogeneous factorization)

Proposition 4 ([GX2017,Corrollary 2.11])

Let Γ be a $(G, 2)$-arc-transitive digraph, M be a vertex-transitive normal subgroup of G. Then Γ is M-arc-transitive.

Let G be a transitive permutation group acting on Ω and let $v \in \Omega$. Then a G_v-orbit on Ω are called *G-suborbit* relative to v. For a G-suborbit w^{G_v}, if $(v,w)^G = (w,v)^G$, then w^{G_v} is said to be *self-paired*, otherwise *non-self-paired*.

Note that a G-arc-transitive digraph arises from a non-self-paired G-suborbit, equivalently, arises from a (G_v, G_v)-double coset $G_v g G_v$ with $g^{-1} \notin G_v g G_v$.

Lemma 1

Let Γ be a $(G,2)$-arc-transitive digraph with arc (v,w). Let $g \in G$ such that $v^g = w$. Then w is in a non-self-paired G-suborbit and $g^{-1} \notin G_v g G_v$.

Let $G = PSL_3(3).2$, $H = A\Gamma L_1(9)$. Find representations of (H, H)-double cosets in G, and check whether $g^{-1} \in HgH$ for each representation g.

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G:=AutomorphismGroupSimpleGroup("L",3,3);
Gas:=Subgroups(G:OrderEqual:=9*8*2);
Ga:=Gas[1]`subgroup;
gs,_:=DoubleCosetRepresentatives(G, Ga, Ga);
g:=gs[1];
Gab:=Ga meet Ga^g;
Tr:=Transversal(Ga,Gab);
Tr1:=[t:t in Tr| g*t*g  in Ga];
if #Tr1 ge 1 then
print "the suborbit is self-paired";
else
print "the suborbit is non-self-paired";
end if;
```

The almost simple case

From now on, we consider the case G is almost simple.

Hypothesis

Let Γ be a connected G-vertex-primitive $(G, 2)$-arc-transitive digraph of valency at least 3, where G is almost simple with socle T. Take an arc $u \to v$ of Γ. Let g be an element of G such that $u^g = v$ and let $w = v^g$. Then $u \to v \to w$ is a 2-arc in Γ.

We have seen that:

- $|G_v : G_{vw}| \geq 3$, and
- $g^{-1} \notin G_v g G_v$, or equivalently, Γ arises from a non-self-paired G-suborbit.
- Γ is T-arc-transitive, and so Γ also arises from a non-self-paired T-suborbit.
- $G_v = G_{uv} G_{vw}$ is a homogeneous factorization and $G_{uv}^g = G_{vw}$. (In particular, G_{uv} is not conjugate to G_{vw} in G_v.)

The arc-transitivity of T

Why consider the arc-transitivity of T?
Note that $T \leq G \leq \mathrm{Aut}(T)$. There may be many candidates for G. If T has good property, then we only need to do computation in T, not in all candidates for G.

Lemma 2

Γ is T-arc-transitive. Let t be the orbits of T_{uv} on $\Gamma(v)^+$ and $o = |G|/|T|$.

(1) $t = |T_v|/|T_{uv}T_{vw}|$ and t divides o;

(2) If $T_v = T_{uv}T_{vw}$, then Γ is $(T, 2)$-arc-transitive;

(3) If $|T_v| \geq o$, then T_{uv} is not conjugate to T_{vw} in T_v.

(Note that T_{uv} is a normal subgroup of G_{uv} and G_{uv} is transitive on $\Gamma(v)^+$. Consider the action of T_{uv} on $\Gamma(v)^+$.)

nonsolvable composition factor of G_v

Let $\mathbf{R}(G_v)$ be the largest solvable normal subgroup of G_v. Set

$$\overline{G_v} = G_v/\mathbf{R}(G_v),\ \overline{G_{uv}} = G_{uv}\mathbf{R}(G_v)/\mathbf{R}(G_v),\ \text{and}\ \overline{G_{vw}} = G_{vw}\mathbf{R}(G_v)/\mathbf{R}(G_v).$$

Then

- From the factorization $G_v = G_{uv}G_{vw}$, we have $\overline{G_v} = \overline{G_{uv}G_{vw}}$.
- $\overline{G_{uv}}$ and $\overline{G_{vw}}$ has the same nonsolvable composition factors (count with multiplicities). (In particular, both solvable or both nonsolvable).
- If G_v has only one nonsolvable composition factor, then $\overline{G_v}$ is almost simple.

Factorization of almost simple groups: two factors are solvable

For an integer n, we use $\pi(n)$ be the set of primes dividing n; for a group G, we use $\pi(G)$ be the set of primes dividing the order of G.
For convenience, we define two sets of simple groups

$$\begin{aligned} \mathcal{T}_1 &:= \{A_6, M_{12}, \mathrm{Sp}_4(q)(q \text{ even}), \mathrm{P\Omega}_8^+(q)\}. \\ \mathcal{T}_2 &:= \{\mathrm{PSL}_2(q), \mathrm{PSL}_3(3), \mathrm{PSL}_3(4), \mathrm{PSL}_3(8), \mathrm{PSU}_3(8), \mathrm{PSU}_4(2)\}. \end{aligned} \tag{1}$$

Proposition 6

Let H be an almost simple group with socle M. Suppose $M = KL$, where K and L are solvable and $M \not\leq K, L$, then (H, K, L) are determined by [8,Proposition 4.1], in particular, $M \in \mathcal{T}_2$ and $\pi(K) \neq \pi(L)$.

[8] C.H. Li, B.Z. Xia, Factorizations of almost simple groups with a solvable factor, and Cayley graphs of solvable groups, arXiv :1408.0350.

PROPOSITION 4.1. *Let G be an almost simple group with socle L. If $G = HK$ for solvable subgroups H, K of G, then interchanging H and K if necessary, one of the following holds.*

(a) $L = \mathrm{PSL}_2(q)$, $H \cap L \leqslant \mathrm{D}_{2(q+1)/d}$ *and* $q \trianglelefteq K \cap L \leqslant q{:}((q-1)/d)$, *where q is a prime power and* $d = (2, q-1)$.

(b) *L is one of the groups:* $\mathrm{PSL}_2(7) \cong \mathrm{PSL}_3(2)$, $\mathrm{PSL}_2(11)$, $\mathrm{PSL}_3(3)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_3(8)$, $\mathrm{PSU}_3(8)$, $\mathrm{PSU}_4(2) \cong \mathrm{PSp}_4(3)$ *and* M_{11}; *moreover, (G, H, K) lies in* Table 4.1.

Conversely, for each prime power q there exists a factorization $G = HK$ satisfying part (a) *with* $\mathrm{soc}(G) = L = \mathrm{PSL}_2(q)$, *and each triple (G, H, K) in* Table 4.1 *gives a factorization $G = HK$.*

TABLE 4.1.

row	G	H	K
1	$\mathrm{PSL}_2(7).\mathcal{O}$	$7{:}\mathcal{O}$, $7{:}(3 \times \mathcal{O})$	S_4
2	$\mathrm{PSL}_2(11).\mathcal{O}$	$11{:}(5 \times \mathcal{O}_1)$	$\mathrm{A}_4.\mathcal{O}_2$
3	$\mathrm{PSL}_2(23).\mathcal{O}$	$23{:}(11 \times \mathcal{O})$	S_4
4	$\mathrm{PSL}_3(3).\mathcal{O}$	$13{:}\mathcal{O}$, $13{:}(3 \times \mathcal{O})$	$3^2{:}2.\mathrm{S}_4$
5	$\mathrm{PSL}_3(3).\mathcal{O}$	$13{:}(3 \times \mathcal{O})$	$\mathrm{A\Gamma L}_1(9)$
6	$\mathrm{PSL}_3(4).(\mathrm{S}_3 \times \mathcal{O})$	$7{:}(3 \times \mathcal{O}).\mathrm{S}_3$	$2^4{:}(3 \times \mathrm{D}_{10}).2$
7	$\mathrm{PSL}_3(8).(3 \times \mathcal{O})$	$73{:}(9 \times \mathcal{O}_1)$	$2^{3+6}{:}7^2{:}(3 \times \mathcal{O}_2)$
8	$\mathrm{PSU}_3(8).3^2.\mathcal{O}$	$57{:}9.\mathcal{O}_1$	$2^{3+6}{:}(63{:}3).\mathcal{O}_2$
9	$\mathrm{PSU}_4(2).\mathcal{O}$	$2^4{:}5$	$3^{1+2}_+{:}2.(\mathrm{A}_4.\mathcal{O})$
10	$\mathrm{PSU}_4(2).\mathcal{O}$	$2^4{:}\mathrm{D}_{10}.\mathcal{O}_1$	$3^{1+2}_+{:}2.(\mathrm{A}_4.\mathcal{O}_2)$
11	$\mathrm{PSU}_4(2).2$	$2^4{:}5{:}4$	$3^{1+2}_+{:}\mathrm{S}_3$, $3^3{:}(\mathrm{S}_3 \times \mathcal{O})$, $3^3{:}(\mathrm{A}_4 \times 2)$, $3^3{:}(\mathrm{S}_4 \times \mathcal{O})$
12	M_{11}	$11{:}5$	$\mathrm{M}_9.2$

where $\mathcal{O} \leqslant \mathrm{C}_2$, and $\mathcal{O}_1, \mathcal{O}_2$ are subgroups of $\mathcal{O}$ such that $\mathcal{O} = \mathcal{O}_1\mathcal{O}_2$.

Factorization of almost simple groups: two factors have same nonsolvable composition factor

Lemma 3

Let H be an almost simple group with socle M. Suppose $H = KL$ with nonsolvable core-free subgroups K and L such that K and L have the same nonsolvable composition factors and the same multiplicities. Then $H = (K \cap M)(L \cap M)$ with $M \in \mathcal{T}_1$, and interchanging K and L if necessary, one of the following holds:

(1) $M = \mathrm{A}_6$, $(H, K, L) \cong (\mathrm{A}_6, \mathrm{A}_5, \mathrm{A}_5)$ or $(\mathrm{S}_6, \mathrm{S}_5, \mathrm{S}_5)$.

(2) $M = M_{12}$, $(H, K, L) \cong (M_{12}, M_{11}, M_{11})$.

(3) $M = \mathrm{Sp}_4(q)$ with $q \geq 4$ even, $H \leq \mathrm{P\Gamma Sp}_4(q)$, and $(K \cap M, L \cap M) \cong (\mathrm{Sp}_2(q^2).2, \mathrm{Sp}_2(q^2).2)$ or $(\mathrm{Sp}_2(q^2).2, \mathrm{Sp}_2(q^2))$.

(4) $M = \mathrm{P\Omega}_8^+(q)$, $H \leq \mathrm{P\Gamma O}_8^+(q)$, and $(K \cap M, L \cap M) \cong (\mathrm{P\Omega}_7(q), \mathrm{P\Omega}_7(q))$.

(It is easy to prove by using the result of [9-11].)

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[9] C.H. Li, B. Xia, Factorizations of almost simple groups with a factor having many nonsolvable composition factors. Journal of Algebra 528 (2019) 439-473.

[10] C.H. Li, L. Wang, B. Xia, The exact factorizations of almost simple groups, arxiv 2012.09551v2.

[11] R.W. Baddeley, C.E. Praeger, On classifying all full factorisations and multiple-factorisations of the finite almost simple groups. Journal of Algebra 204 (1998) 129-187.

The case G_v has only one nonsolvable composition factor

For a group X, let $X^{(\infty)}$ be the smallest normal subgroup of X such that $X/X^{(\infty)}$ is soluble.

A group X is called quasisimple if $X = X'$ and $X/\mathbf{Z}(X)$ is simple. Note that X is the unique subgroup of X which has a composition factor isomorphic to $X/\mathbf{Z}(X)$. (If $Y < X$ has a composition factor isomorphic to $X/\mathbf{Z}(X)$, then $Y' = (Y\mathbf{Z}(X))' = X' = X$, a contradiction.)

$$\mathcal{T}_1 := \{\mathrm{A}_6, M_{12}, \mathrm{Sp}_4(q)(q \text{ even}), \mathrm{P}\Omega_8^+(q)\}.$$
$$\mathcal{T}_2 := \{\mathrm{PSL}_2(q), \mathrm{PSL}_3(3), \mathrm{PSL}_3(4), \mathrm{PSL}_3(8), \mathrm{PSU}_3(8), \mathrm{PSU}_4(2)\}.$$

Lemma 4

Suppose that G_v has only one nonsolvable composition factor M.

(1) If G_v is almost simple, then Γ is $(T, 2)$-arc-transitive, and (G_v, G_{uv}, G_{vw}) satisfies (1)-(4) of Lemma 3.

(2) If $M \notin \mathcal{T}_1 \cup \mathcal{T}_2$, then both G_{uv} and G_{vw} have a nonsolvable composition factor M and $G_v^{(\infty)}$ is not quasisimple.

(3) If $G_v^{(\infty)}$ is quasisimple, then $M \in \mathcal{T}_1 \cup \mathcal{T}_2$, and the factorization $\overline{G_v} = \overline{G_{uv}G_{vw}}$ satisfies Lemma 3 or Proposition 6.

(1) was proved in [GLX2019,Corollary 3.4]. (2) and (3) was proved similarly. The key is Proposition 2, that is, each nontrivial normal subgroup of G_v is not normalized by g.

I use an examples to illustrate the proof of (2).
Let $G = Co_1$, $G_v = 3.Suz{:}2$.

- Then $G_v^{(\infty)} = 3.Suz$ and $\overline{G_v} = Suz{:}2$.
- Since $Suz \notin \mathcal{T}_1 \cap \mathcal{T}_2$, both $\overline{G_{uv}}$ and $\overline{G_{vw}}$ contains a composition factor Suz, so do G_{uv} and G_{vw}.
- Note that $G_{uv}^{(\infty)} \leq G_{uv} \cap G_v^{(\infty)} \leq G_v^{(\infty)}$ as $G_{uv}/(G_{uv} \cap G_v^{(\infty)})$ is solvable and $G_{uv}^{(\infty)}$ is the smallest normal subgroup N of G_{uv} such that G_{uv}/N is solvable.
- Since $G_v^{(\infty)}$ is quasisimple, $G_{uv}^{(\infty)} = G_v^{(\infty)}$. Similarly, $G_{vw}^{(\infty)} = G_v^{(\infty)}$.
- Let $Y = (G_{uv}^{(\infty)})^g$. Then $Y/(Y \cap G_{vw}^{(\infty)}) \cong YG_{vw}^{(\infty)}/G_{vw}^{(\infty)} \leq G_{vw}/G_{vw}^{(\infty)}$ is solvable. Thus $Y \cap G_{vw}^{(\infty)}$ has a composition factor Suz.
- Since $G_{vw}^{(\infty)}$ is quasisimple, $Y \cap G_{vw}^{(\infty)} = G_{vw}^{(\infty)}$ and so $Y = G_{vw}^{(\infty)}$, that is $(G_{uv}^{(\infty)})^g = G_{vw}^{(\infty)} = G_v^{(\infty)}$, which contradicts Proposition 2.

Computation method

The homogeneous factorization $G_v = G_{uv}G_{vw}$ implies G_{uv} and G_{vw} are not conjugate in G_v but in G, and

$$|G_v||G_{vu} \cap G_{vw}| = |G_{uv}||G_{vw}| = |G_{uv}|^2.$$

- We use Magma commands **AutomorphismGroupSimpleGroup** and **MaximalSubgroups** to construct G and G_v.
- Let $|G_v| = p_1^{s_1}...p_t^{s_t}$. Then $|G_{vw}|$ is multiple of $p_1^{\lfloor \frac{s_1}{2} \rfloor}..p_t^{\lfloor \frac{s_t}{2} \rfloor}$. We use command **Subgroups(Gv :OrderMultipleOf:=m)** can compute all possibilities of G_{uv} and G_{vw}. Then check whether $|G_v||G_{vu} \cap G_{vw}| = |G_{uv}||G_{vw}|$.
- The step finding all candidates of G_{uv} and G_{vw} can be optimized when G_v has only one nonsolvable composition factors, or when G_v has a normal p-subgroup N such that G_v/N is small.

For example, Let $G = Th$.

(1) Let $G_v := 2^5.\mathrm{PSL}_5(2)$. Consider the factorization $\overline{G_v} = \overline{G_{uv}G_{vw}}$, we have both G_{uv} and G_{vw} have a composition factor $\mathrm{PSL}_5(2)$. Note that $|\mathrm{PSL}_5(2)| = 2^{10}{\cdot}3^2{\cdot}5{\cdot}7{\cdot}31$. So $|G_{uv}| = |G_{vw}|$ is multiple of $2^{13}{\cdot}3^2{\cdot}5{\cdot}7{\cdot}31$.

(2) Let $G_v := 3.[3^8].2\mathrm{S}_4$. Now G has a normal p-subgroup $N = 3.[3^8]$ such that $G_v/N \cong 2\mathrm{S}_4$. Note that $G_v/N \cong (G_{uv}N/N)(G_{vw}N/N)$. Since $G_{uv} \cong G_{vw}$ and N is a 3-group, a Sylow 2-subgroup of $G_{uv}N/N$ is isomorphic to $G_{vw}N/N$'s. By computing all factorizations of G_v/N satisfying that two factors have isomorphic Sylow 2-subgroups, we obtain $G_{uv}N/N = G_{vw}N/N = G_v/N \cong 2\mathrm{S}_4$. Then $|G_{uv}| = |G_{vw}|$ is multiple of $2^4{\cdot}3^5$.

- When T is a simple classical group of Lie type of large order, we may do computation only in T. The command **ClassicalMaximals** is used to constructed the preimage of T_v in the quasisimple group relative to T.
- However, the above MAGMA computational method is not feasible if $|G|$ and $|G : G_v|$ are very large, for example, T is a simpe group of Lie type and T_v is a parabolic subgroup. This case is dealt with by considering the suborbits of T.

$T = \mathrm{A}_n$

By the classification of maximal subgroups of alternating and symmetric groups, G_v satisfies one of the following:

(1) $G_v = (\mathrm{S}_m \times \mathrm{S}_k) \cap G$, where $n = m + k$, $m > k$ (intransitive case);

(2) $G_v = (\mathrm{S}_m \wr \mathrm{S}_k) \cap G$, where $n = mk$, $m > 1$ and $k > 1$ (imprimitive case);

(3) $G_v = \mathrm{AGL}(k, p) \cap G$, where $n = p^k$ (affine case);

(4) $G_v = (S^k : (\mathrm{Out}(S) \times \mathrm{S}_k)) \cap G$ where S is a nonabeilain simple group, $k \geq 2$ and $n = |S|^{k-1}$ (diagonal case);

(5) $G_v = (\mathrm{S}_m \times \mathrm{S}_k) \cap G$, where $n = m^k$, $m \geq 5$ and $k \geq 2$ (primitive wreath case);

(6) $S \leq G_v \leq \mathrm{Aut}(S)$, where S is a nonabelian simple group, $S \neq \mathrm{A}_n$ and G_v acts primitively on $\{1, 2, ..., n\}$ (almost simple case).

The case (1) and (3) are impossible by PWY2020. If (4) or (5) happens, then $|V(\Gamma)| > 30758154560$.

almost simple case

Let $\mathcal{N} := 30758154560$. Suppose G_v is of almost simple case, that is, $G_v \neq \mathrm{A}_n$ is almost simple and H acts primitively on $\{1, 2, ..., n\}$.

- Now $\mathrm{Soc}(G_v) \in \{\mathrm{A}_6, M_{12}, \mathrm{PSp}_4(2^f), \mathrm{P}\Omega_8^+(q)\}$.
- From the upper bound for the order of primitive group given in [12,Theorem 1.1] , we have $|G_v| < n^{1+[\log_2(n)]}$ (with some exceptions). Then $\frac{n!}{n^{1+[\log_2(n)]}} \leq \mathcal{N}$ implies $n \leq 20$.
- $\mathrm{PSp}_4(2^f), \mathrm{P}\Omega_8^+(q)$ has no primitive permutation representation of degree less than 20.
- If $G_v = \mathrm{A}_6$, then $n = 6$ or 15 and $G_{uv} \cong G_{vw} \cong \mathrm{A}_5$. When $n = 6$, G_{uv} is not conjugate to G_{vw} in G because one is transitive on $\{1, ..., 6\}$ while the other is not. When $n = 15$, G_{uv} is not conjugate to G_{vw} in G by Magma.
- If $G_v = M_{12}$, then $n = 12$ and $G_{uv} \cong G_{vw} \cong M_{11}$. K and L are not conjugate in T.

[12] A. Maróti, On the orders of primitive groups. Journal of Algebra 258 (2002) 631-640.

imprimitive case

Suppose G_v is of imprimitive case, that is, $G_v = (\mathrm{S}_m \wr \mathrm{S}_k) \cap G$, where $n = mk$, $m > 1$ and $k > 1$. Then $|V(\Gamma)| = |\mathrm{S}_n|/|\mathrm{S}_m \wr \mathrm{S}_k| = (mk)!/(m!)^k k!$.

- By computation with computer, the (m, k) such that $V(\Gamma) \leq \mathcal{N}$ are

$$(3 \leq m \leq 19, 2), (2 \leq m \leq 9, 3), (2 \leq m \leq 5, 4),$$
$$(2 \leq m \leq 4, 5), (3, 6 \leq k \leq 7), (2, 6 \leq k \leq 11).$$

- The case $(2 \leq m \leq 4, 5)$, $(3, 6 \leq k \leq 7)$ and $(2, 6 \leq k \leq 11)$ can be direct ruled out by computation in Magma.
- The case $k = 2$ with $m \geq 11$, and the case $k = 3$ and $m \geq 7$ is difficult. (The reason is G_v is solvable, leading that G_v has many subgroups of order multiple of $p_1^{\lfloor \frac{s_1}{2} \rfloor} .. p_t^{\lfloor \frac{s_t}{2} \rfloor}$.)
  So we consider the suborbits of T.

suborbits of G with G_v an imprimitive wreath product group

For two elements v, w in Ω, where $v = \{V_1, V_2, ..., V_k\}$ and $w = \{W_1, W_2, ..., W_k\}$, we have

$$V_i = \bigcup_{1\leq j\leq k} (V_i \cap W_j), \text{ and } \sum_{1\leq j\leq k} |V_i \cap W_j| = m.$$

$$W_j = \bigcup_{1\leq i\leq k} (V_i \cap W_j), \text{ and } \sum_{1\leq i\leq k} |V_i \cap W_j| = m.$$

We say the matrix $M(v, w) := [|V_i \cap W_j|]_{k\times k}$ is a representation of the intersection of v and w. Note that the intersection of v and w may has many representations (if changing the order of V_i and W_j we may obtain a different representation), but all representations form the next set

$$\{P_1 M(v, w) P_2 | P_1, P_2 \text{ are } k\times k \text{ permutation matrix}\}.$$

(Recall a $k\times k$ *permutation matrix* is a matrix obtained by permuting the rows of an $k\times k$ identity matrix according to some permutation on $\{1, 2, ..., k\}$.)

Lemma 5

Let $G = \mathrm{S}_n$, $T = \mathrm{A}_n$, Ω the set of imprimitivity partitions of $\{1, 2, ..., n\}$ with k blocks of size m, $v, w \in \Omega$, and let $M(v, w)$ be a representation of the intersection of v and w. Then

(1) $w' \in w^{G_v}$ if and only if $M(v, w') = P_1 M(v, w) P_2$, where P_1, P_2 are two $k \times k$ permutation matrices.

(2) v, w are interchanged by $g \in G$ if and only if $M(v, w)^{\mathrm{T}} = P_1 M(v, w) P_2$, where $M(v, w)^{\mathrm{T}}$ is the transpose of $M(v, w)$ and P_1, P_2 are two $k \times k$ permutation matrices.

(3) Suppose that v, w are interchanged by an odd permutation $g \in G$. Then v, w are interchanged by some $t \in T$ if and only if G_{vw} contains odd permutations.

imprimitive case

Suppose G_v is of imprimitive case, that is, $G_v = (\mathrm{S}_m \wr \mathrm{S}_k) \cap G$, where $n = mk$, $m > 1$ and $k > 1$. Then $|V(\Gamma)| = |\mathrm{S}_n|/|\mathrm{S}_m \wr \mathrm{S}_k| = (mk)!/(m!)^k k!$.

- By computation, we remain the case $k = 2$ with $m \geq 11$, and the case $k = 3$ and $m \geq 7$

Apply Lemma 5.

- If $k = 2$, then M_{vw} is always symmetric and hence all suborbits of T are self-paired.
- If $k = 3$, then $2 \leq m \leq 9$. By computation, the non-self-paired orbital cases are:

(1) $m = 6$, and $M_{vw} = \begin{bmatrix} 0 & 2 & 4 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ or its transpose;

(2) $m = 7$, and $M_{vw} = \begin{bmatrix} 0 & 2 & 5 \\ 3 & 3 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ or its transpose;

(3) $m = 8$, and $M_{vw} = \begin{bmatrix} 0 & 3 & 5 \\ 4 & 2 & 2 \\ 4 & 3 & 1 \end{bmatrix}$ or its transpose.

For the case $m = 6$, and $M_{vw} = \begin{bmatrix} 0 & 2 & 4 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$:

- Now, $n = 18$, $G_v = \mathrm{S}_6 \wr \mathrm{S}_3 \cap G$, $G_{vw} = (\mathrm{S}_3^2 \times \mathrm{S}_2^3 \times \mathrm{S}_4){:}\mathrm{S}_2 \cap G$.
- Let $N = \mathrm{S}_m^k \cap G$ be the base group of G_v, then $N \trianglelefteq G_v$ and $G_v/N \cong \mathrm{S}_n$ (This is clear when $G = \mathrm{S}_n$. If $G = \mathrm{A}_n$, then N is with index 2 in S_m^k as S_m^k contains odd permutations, so $|G_v/N| = |\mathrm{S}_k|$ and hence $G_v/N \cong \mathrm{S}_k$).
- In the factorization $\mathrm{S}_k \cong G_v/N = (G_{uv}N/N)(G_{vw}N/N)$, one factor should be transitive (on k points) (see [13,1.3,1.4] for a proof).
- Now $G_{vw}N/N \cong \mathrm{S}_2$ and $G_{uv}N/N \cong 1$, a contradiction. (A subgroup of $G_v/N = \mathrm{S}_3$ describes the symmetry of rows. In M_{vw}, the second and third row can be interchanged, so $G_{vw}N/N \cong \mathrm{S}_2$. While for G_{uv}, we have $M_{vu} = \begin{bmatrix} 0 & 3 & 3 \\ 2 & 2 & 2 \\ 4 & 1 & 1 \end{bmatrix}$, there is no pair of rows can be interchanged, this means $G_{uv} \leq N$ and $G_{uv}N/N \cong 1$.)

[13] J. Wiegold and A.G. Williamson, The factorization of the alternating and symmetric groups, Math. Z. 175 (1980), 171-179.

T is a simple sporadic group

Lemma 6

Suppose that Hypothesis holds, then T is not one of the next 22 groups:

$$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, HS, J_2, McL, Suz, J_3, Co_3, Co_2, He, Fi_{22}, Ru.$$
$$Th, Fi_{23}, J_4, Ly, HN, O'N.$$

- If T is one group in the first row, then G has a permutation representation of small degree (less than 10000). We can obtain G and G_v in MAGMA using its commands, then computation shows that G_v has no homogeneous factorization $G_v = KL$ such that $|G_v : K| \geq 3$ and K and L are conjugate in G.
- Let T be one group in the second row. If $T = Fi_{23}$, then we can construct G and G_v using the MAGMA commands. For other groups, we can construct G and G_v using the information of generators given in the Web Atlas [14].

[14] R. A. Wilson, S. J. Nickerson, J. N. Bray et al., An Atlas of Group Representations, ver. 3, http://brauer.maths.qmul.ac.uk/Atlas/v3/.

Note that the information of generators for maximal subgroups $2^{3+2+6}.(3\times \mathrm{PSL}_3(2))$, $3^4{:}2.(\mathrm{A}_4\times\mathrm{A}_4).4$ and $(\mathrm{A}_6\times\mathrm{A}_6).\mathrm{D}_8$ of HN is not given in the Web Atlas.

- The groups $2^{3+2+6}.(3\times \mathrm{PSL}_3(2))$ and $3^4{:}2.(\mathrm{A}_4\times\mathrm{A}_4).4$ can be constructed using the method as in [15, p.318]. For example, to construct $2^{3+2+6}.(3\times \mathrm{PSL}_3(2))$, we can first construct a Sylow 2-subgroup P of HN (note that $|HN|_2 = |2^{3+2+6}.(3\times \mathrm{PSL}_3(2))|_2 = 2^{14}$), then compute all normal 2-subgroups of order 2^{11} of P, and their normalizer in HN; if the normalizer has order $|2^{3+2+6}.(3\times \mathrm{PSL}_3(2))|$, then the normalizer is the required maximal group $2^{3+2+6}.(3\times \mathrm{PSL}_3(2))$.
- The group $(\mathrm{A}_6\times\mathrm{A}_6).\mathrm{D}_8$ has a normal subgroup $(\mathrm{A}_6\times\mathrm{A}_6).2^2$ contained in A_{12} (the information of generators of A_{12} is given in Web Atlas), so we can construct the group $(\mathrm{A}_6\times\mathrm{A}_6)^{\cdot}2^2$ in A_{12} first and then compute its normalizer, which is the required $(\mathrm{A}_6\times\mathrm{A}_6).\mathrm{D}_8$.

[15] T.C. Burness, E.A. O'Brien, R.A. Wilson, Base sizes for sporadic simple groups. Israel J. Math. 177 (2010) 307-333.

- If G_v has only one nonsolvable composition factor and $G_v^{(\infty)}$ is quasisimple, then we may apply Lemma 4 to rule out this candidate directly. For example $G = Th$ and G_v is ${}^3D_4(2){:}3$, or $\mathrm{PSU}_3(8){:}6$, or $\mathrm{PSL}_2(19){:}2$, or $\mathrm{PSL}_3(3)$, or M_{10} (note that M_{10} is not a subgroup of S_6), or S_5, or $(3\times G_2(3)){:}2$.
- If G_v is a metacyclic Frobenius group, then we may apply Proposition 2 to rule out it. For example, $G = Th$ and $G_v = 31{:}15$. Now from the homogeneous factorization $G_v = G_{uv}G_{vw}$ we see both G_{uv} and G_{vw} contains the normal subgroup $M \cong \mathrm{C}_{31}$ of G_v. Since G_v has only one subgroup isomorphic to C_{31}, $M^g = M$, which contradicts Proposition 2.

There are two cases is difficult to compute.

- $G = Fi_{23}$ and $G_v = 3^{1+8}.2^{1+6}.3^{1+2}.2\mathrm{S}_4$. Then $|G_{uv}|$ is multiple of $2^6{\cdot}3^7$ as $|G_v| = 2^{11}{\cdot}3^{13}$. It is difficult to compute all subgroups of order multiple of $2^6{\cdot}3^7$ in G_v. So we take $N = 3^{1+8}$ and consider the factorizations of $2^{1+6}.3^{1+2}.2\mathrm{S}_4 \cong G_v/N = (G_{uv}N/N)(G_{vw}N/N)$, where $G_{uv}N/N, G_{vw}N/N$ have order multiple of 2^6. Since $G_{uv} \cong G_{vw}$, $G_{uv}/O_3(G_{uv}) \cong G_{vw}/O_3(G_{vw})$. Note that $G_{uv} \cap N \leq O_3(G_{uv})$ and $G_{vw} \cap N \leq O_3(G_{vw})$. Thus $G_{uv}N/N \cong G_{uv}/(G_{uv} \cap N)$ has a normal 3-subgroup M_1 isomorphic $O_3(G_{uv})/(G_{uv} \cap N)$, and $G_{vw}N/N \cong G_{vw}/(G_{vw} \cap N)$ has a normal 3-subgroup M_2 isomorphic $O_3(G_{vw})/(G_{vw} \cap N)$ such that $(G_{uv}N/N)/M_1 \cong (G_{vw}N/N)/M_2$. By computing all factorizations of $2^{1+6}.3^{1+2}.2\mathrm{S}_4$, we find no desired factorization. Therefore this case is impossible.

- $G = HN{:}2$ and $G_v = (\mathrm{S}_6 \times \mathrm{S}_6).2^2$. Computation shows that G_v indeed has homogeneous factorization $G_v = KL$, where $K \cong L$ and $|G_v : K| \geq 3$. However, some computation evidences show that K and L is not conjugate in G. We do as follows. The groups G and G_v are constructed by 133×133 matrices over $\mathbb{F}_5$ (note that the minimal degree of permutation representation of G is 1140000 and it is too large for computation.) Computation shows that G_v has 10 homogeneous factorizations $G_v = KL$, 8 of them with $|K| = 1440$ and the other 2 with $|L| = 2880$. The difficulty is that the MAGMA command **IsConjugate** is not valid when checking whether K and L. So we compute the conjugacy classes of K and L to check whether K and L are conjugate in G. In a homogeneous factorization $G_v = KL$, we find there is an element of order 2 in L such that it is not similar to any element of order 2 in L (using the MAGMA command **IsSimilar**). This implies K and L is not conjugate in G as $G \leq \mathrm{GL}_{133}(5)$. For other homogeneous factorizations, we also find such element. Therefore this case is impossible.

Lemma 7

Suppose that Hypothesis holds and suppose that $|V\Gamma| \le \mathcal{N}$. Then T is not one of $\mathbb{M}, \mathbb{B}, Fi'_{24}, Co_1$.

- The Monster group $\mathbb{M}$ has no maximal subgroup of index no more than $\mathcal{N}$.
- If $T = \mathbb{B}$, then $G = T$ and $G_v = 2.{}^2E_6(2){:}2$.
- If $T = Fi'_{24}$, then $G = T$ and $G_v = Fi_{23},\ 2^{\cdot}Fi_{22}{:}2,\ (3{\times}\mathrm{P}\Omega_8^+(3){:}3){:}2$. (Note that if $T_v = (3{\times}\mathrm{P}\Omega_8^+(3){:}3){:}2$, then $\overline{G_v} = \mathrm{P}\Omega_8^+(3){:}\mathrm{S}_3$ contains a graph automorphism of order 3 and hence $\overline{T_v}$ is not a subgroup of $\mathrm{P\Gamma O}_8^+(q)$. This can be verified by Magma).
- If $T = Co_1$, then $G = T$ and the possible G_v are

  $Co_2, 3.Suz.2, Co_3, \mathrm{PSU}_6(2){:}\mathrm{S}_3, (\mathrm{A}_4{\times}G_2(4)){:}2,$

  $2^{1+8}.\mathrm{P}\Omega_8^+(2), 2^{11}{:}M_{24}, 2^{2+12}{:}(\mathrm{A}_8{\times}\mathrm{S}_3), 2^{4+12}.(\mathrm{S}_3{\times}3.\mathrm{S}_6), 3^2.\mathrm{PSU}_4(3).\mathrm{D}_8, 3^6{:}2.M_{12}.$

T is a simple group of Lie type

Some helpful results:

- Alavi and Burness [16] determined all large maximal subgroups of finite simple groups and their automorphism groups. Formally, a subgroup X is called *large* in group Y if $|X| > |Y|^{1/3}$. Note that if $|T|^{2/3} > \mathcal{N}$, then our assumption $|V(\Gamma)| = |T : T_v| \leq \mathcal{N}$ implies that T_v is large in T.
- Maximal parabolic subgroups are always subgroups of T with small index. So we need often considering the case that T_v is a parabolic subgroup. If T_v is a parabolic subgroup, then the information of T-suborbits can be computed by computing on the Weyl group of T and roots of T. See [17, Chapter 2].
- Degree of the minimal permutation representation of T, see [18,Table 4]).

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[16] S.H. Alavi, T.C. Burness, Large subgroups of simple groups. J. Algebra 421 (2015) 187-233.

[17] R.W. Carter, Finite Groups of Lie Type: Conjugacy Classes and Complex Characters. John Wiley & Sons, New York, 1985.

[18] S. Guest, J. Morris, C.E. Praeger, P. Spiga, On the maximum orders of elements of finite almost simple groups and primitive permutation groups. Transactions of the American Mathematical Society 367 (2015) 7665-7694.

Lemma 8

Suppose that T is a simple group of Lie type such that $|T|^{2/3} \leq \mathcal{N}$ and $T \neq \mathrm{PSL}_2(q)$. Then T is one of the following groups:

(1) $\mathrm{PSL}_n(q)$, or $\mathrm{PSU}_n(q)$, where (n, q) is $(3, q \leq 97)$, $(4, q \leq 11)$, $(5, q \leq 4)$, $(6, 2)$, or $(7, 2)$;

(2) $\mathrm{PSp}_n(q)$, where (n, q) is $(4, q \leq 37)$, $(6, q \leq 5)$, or $(8, 2)$;

(3) $\mathrm{P}\Omega_n^{\epsilon}(q)$, where (n, q) is $(7, 3)$, $(7, 5)$, $(8, 2)$, $(8, 3)$, or $(10, 2)$.

(4) ${}^2B_2(2^3), {}^2B_2(2^5), {}^2B_2(2^7), {}^2B_2(2^9)$, ${}^2F_4(2)', {}^2G_2(3^3)$, ${}^3D_4(2)$, ${}^3D_4(3)$, $F_4(2)$, $G_2(q)(q \leq 11)$.

Parabolic subgroup

Let T be a simple group of Lie type in characteristic p. Let U be a Sylow p-subgroup of T.

- The normalizer of U in T is called a *Borel subgroup* of T, and $B = U{:}H$, moreover, H is called the *Cartan subgroup*. ($H \neq 1$.)
- A subgroup P of T is called a *Parabolic subgroup* of T if P contains a conjugate of B.

When P is a standard parabolic subgroup, the method of how to computing the subdegrees of T (acting on $[T : P]$) were introduced in [19].

[19] A.E. Brouwer, A.M. Cohen, Computation of some parameters of Lie geometries, Mathematisch Centrum. ZW, afdeling zuivere wiskunde, 1983, pp. 21.

(B, N)-pair

A pair of subgroups B, N of a group G is called a (B, N)-pair if the following axioms are satisfied:

(1) $G = \langle B, N \rangle$.

(2) $H := B \cap N$ is normal in N.

(3) The group $W := N/H$ (Weyl group) is generated by a set of involutions w_i, $i \in I$.

(4) If $n_i \in N$ maps to w_i under the natural homomorphism N to into W, and if $n \in N$, then

 (i) $Bn_iB.BnB \subseteq Bn_inB \cup BnB$.

 (ii) $n_iBn_i \neq B$.

Some important properties:

- $G = \bigcup_{w \in W} BwB$.
- any element w in W is a product of some w_i, the length of w, saying $l(w)$, is smallest integer $q > 0$ such that w is the product of a sequence of q elements.
- $Bn_iB.BnB = Bn_inB \cup BnB$ if $l(n_i\overline{n}) = l(\overline{n}) - 1$; $Bn_iB.BnB = Bn_inB$ if $l(n_i\overline{n}) = l(\overline{n}) + 1$;

- For each $J \subseteq I$, $P_J := BN_JB$ is a subgroup of G, where $N_J = H\langle n_i : i \in J\rangle$. If $J = \emptyset$, then $B = P_J$. The group P_J is called the standard parabolic subgroup (associated with J).
- For a double coset W_JvW_J in W, there is a unique element $w \in W_JvW_J$ such that w is with shortest length (such w is called (J, J)-reduced), see [17]. Let D_J be the set of (J, J)-reduced elements in W.
- For a coset W_Jv in W, there is a unique element $u \in W_Jv$ such that u is with shortest length (such u is called $(J, \emptyset)$-reduced). Let R_J be the set of $(J, \emptyset)$-reduced elements in W.
- $PwP = BW_JBwBW_JB = BW_JwW_JB$, where $w \in D_J$. As a consequence, each (P_J, P_J)-double coset PwP in G corresponds to a (W_J, W_J)-double coset W_JwW_J in W, and it is a bijection.

Proposition 6 ([19, Proposition 2])

Let notations be as above. Then G (on $[G : P_J]$) has same rank and number of non-self-paired suborbits as its Weyl group W (on $[W : W_J]$). A suborbit of G corresponds to P_JwP_J, where $w \in D_J$, is selfpaired if and only if w is an involution, and its length is $\sum_{v\in R_J\cap wW_J} q^{l(v)}$, where $l(v)$ is the length of v.

computation in Magma

For example, to compute the rank and suborbit of $G = \mathrm{PSL}_5(2)$ acting on $[G : P_J]$, where $P_J = [2^7]{:}\mathrm{PSL}_3(2)$, the stabilizer in G of a $(1, 4)$-flag, we can do as follows

```
W := CoxeterGroup(GrpFPCox,"A4");J:={2,3};
DJ,_:=Transversal(W, J, J);print "the rank:", #DJ;
W0,phi := CoxeterGroup(GrpPermCox, W);
DJ1:=[phi(DJ[i]): i in [2..#DJ]|  Order(DJ[i]) ne 2];
print "number of nonself-paired suborbits:", #DJ1;
RJ:=Transversal(W, J);
WJ:= StandardParabolicSubgroup(W0, J);
for g in DJ do
RJ_wWJ:=[w: w in RJ|g eq TransversalElt(W0,WJ,phi(w),WJ)];
RJ_wWJ1:=[Length(w): w in RJ_wWJ];
RJ_wWJ1; //the set [l(v):v  \in R_J  \cap wW_J]
end for;
```

Let $T_v = P_J$ be a parabolic subgroup of T. Now, we have find all non-self-paired T-suborbits and known their lengths.

- Since $T_v g T_v = P_J g P_J$, where $g \in W$, $T_{vw} = P_J \cap P_J^g$ and $T_{uv} = P_J \cap P_J^{g^{-1}}$.
- If we know $|T_{uvw}| = |P_J \cap P_J^g \cap P_J^{g^{-1}}|$, then we can check whether $|T_v| = t|T_{uv}T_{vw}| = \frac{t|T_{uv}||T_{vw}|}{|T_{uvw}|}$, where $t \mid |G/T|$.
- We can estimate the above equation by $|T_v|_q |T_{uvw}|_q = t_q |T_{uv}|_q^2$.
- The values of $|T_v|_q$ and $|T_{uv}|_q^2$ are known. The value of $|T_{uvw}|_q$ can be estimated by computing on the roots of T.

$|T_v|_q$, $|T_{uv}|_q^2$ and $|T_{uvw}|_q$

The next results can be found or obtained from Carter's book.

- $P_J = \langle H, X_r | r \in \Phi^+ \cup \Phi_J \rangle$, where Φ_J is the set of roots spanned by fundamental roots in J. Moreover, $|P_J|_q = q^{m_1}$, where m_1 is the number of positive roots in $\Phi^+ \cup \Phi_J$.
- $P_J \cap P_J^g = \langle H, X_r | r \in (\Phi^+ \cup \Phi_J) \cap (\Phi^+ \cup \Phi_J)^{g^{-1}} \rangle$. Therefore, $|P_J \cap P_J^g|_q = q^{m_2}$, where m_2 is the number of positive roots in $(\Phi^+ \cup \Phi_J) \cap (\Phi^+ \cup \Phi_J)^{g^{-1}}$.
- $B = \langle H, X_r | r \in \Phi^+ \rangle = UH$. So
  $B \cap B^g = \langle H, X_r | r \in \Phi^+ \cap (\Phi^+)^{g^{-1}} \rangle = (U \cap U^g)H$.
- $B \cap B^g \cap B^{g^{-1}} = \langle H, X_r | r \in \Phi^+ \cap (\Phi^+)^{g^{-1}} \cap (\Phi^+)^g \rangle = (U \cap U^g \cap U^{g^{-1}})H$.
  Therefore, $|P_J \cap P_J^g \cap P_J^{g^{-1}}|_q$ is divisible by q^{m_3}, where m_3 is the number of positive roots in $\Phi^+ \cap (\Phi^+)^{g^{-1}} \cap (\Phi^+)^g$.

An example for $T = G_2(q)$

Let $T = G_2(q)$, $T_v = [q^6]{:}(q-1)^2$ be a Borel subgroup of T, and G contains a graph automorphism of T.

- Computation on the Weyl group of T shows that T has rank 12 and there are 4 non-self-paired suborbits of length q^2 or q^4.
  The Weyl group W of $G_2(q)$ is a dihedral group of order 12. So the rank is 12. In D_{12}, there are 4 elements with order greater than 2. So there are 4 non-self-paired suborbits.)
- If the length is q^4, then $|T_{vw}|_p^2 = q^4 < q^6 = |T_v|_p$, a contradiction.
- Hence the length is q^2 and so $T_{vw} = [q^4]{:}(q-1)^2$. By computing the roots, $T_{uvw} = [q^2]{:}(q-1)^2$. Then $|T_v||T_{uvw}| = |T_{uv}||T_{vw}|$ and so $T_v = T_{uv}T_{vw}$, which means Γ is $(T,2)$-arc-transitive. However, we next show that Γ is not $(G,2)$-arc-transitive.

Let $\Phi = \Phi^+ \cup \Phi^-$ be a root system of T with fundamental roots a, b, and $\Phi^+ = \{b, a, b+a, b+2a, b+3a, 2b+3a\}$. Let H be the Cartan subgroup of T. Take $T_v = \langle H, X_r : r \in \Phi^+\rangle$ be the Borel subgroup. Label roots as follows:

b	a	$b+a$	$b+2a$	$b+3a$	$2b+3a$	$-b$	$-a$	$-b-a$	$-b-2a$	$-b-a$	$-2b-3a$
1	2	3	4	5	6	7	8	9	10	11	12

- $W = \langle w_a, w_b\rangle$, where $w_a = (1,5)(2,8)(3,4)(7,11)(9,10)$ and $w_b = (1,7)(2,3)(5,6)(8,9)(11,12)$. It can be view as a permutation group on Φ.
- G_v contains a graph automorphism γ normalizing T_v. By computation with Magma, $\gamma = (1,2)(3,5)(4,6)(7,8)(9,11)(10,12)$. (Graph automorphism can also be view as permutation group on Φ, see [19, Section 12.4], and it can be contained from the normalizer of W in $\mathrm{Sym}(\Phi)$.)
- By computation, $g = (1,11,12,7,5,6)(2,4,3,8,10,9)$ (or its inverse). Note that $\Phi^+ \cap (\Phi^+)^{g^{-1}} = \{2,4,5,6\}$ as $1^{g^{-1}} = 6$, $6^{g^{-1}} = 5$, $3^{g^{-1}} = 4$ and $4^{g^{-1}} = 2$. So

$$T_{vw} := \langle H, X_r | r \in \{b, b+a, b+2a, 2b+3a\}\rangle.$$

[19] R.W. Carter, Simple groups of Lie type, Wiley, London, 1972.

- Recall G_v contains a graph automorphism γ and $|G_{vw} : T_{vw}| = |G_v : T_v| = |G : T|$. So T_{vw} is normalized by some element $h = \phi\gamma x \in G_{vw} \setminus T_{vw}$, where $x \in T_v = UH$ and ϕ is a field automorphism.
- Let $M = \langle X_r | r \in \{b, b+a, b+2a, 2b+3a\}\rangle$. Then M is the largest normal p-subgroup of T_{vw}, and hence $M = M^h = M^{\phi\gamma x} = M^{\gamma x}$ (as ϕ normalizes each root subgroup)
- Now $M^\gamma = \langle X_r | r \in \{a, b+3a, b+2a, 2b+3a\}\rangle$ as $\{1,3,4,6\}^\gamma = \{2,5,4,6\}$.
- Since H normalizes each root subgroup, x can be taken as an element in $U = \langle X_r | r \in \Phi^+\rangle$. By the structure of U, U has a normal subgroup $N := \langle X_{2b+3a}, X_{b+3a}\rangle$.
- Since $M = (M^\gamma)^x$, $M/N = (M^\gamma/N)^x$. However, $|M/N| = q^3$ while $|M^\gamma/N| = q^2$, a contradiction.

Difficult cases when T_v is a parabolic subgroup

- $T = G_2(q)$, T_v is a Borel subgroup.
- $T = \mathrm{PSp}_4(q)$, T_v is a Borel subgroup.
- $T = \mathrm{P}\Omega_8^+(q)$, T_v is of type A_1.

Difficult cases when T_v is not a parabolic subgroup

- $T = G_2(q)$,
  (i) $T_v = \mathrm{SL}_2(q)\times\mathrm{SL}_2(q)$ with $q \in \{4, 8, 16\}$.
  (ii) $T_v = (\mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)).2$ with $q \in \{3, 5, 7, 9, 11, 13, 17, 19\}$.
- $T = \mathrm{PSU}_n(q)$, T_v is a $\mathcal{C}_5$-subgroup of type $\mathrm{Sp}_n(q)$, where $q \in \{4, 8, 16, 32, 64\}$.
- $T = \mathrm{PSp}_4(q)$, T_v is a $\mathcal{C}_2$-subgroup of type $\mathrm{Sp}_2(q) \wr \mathrm{S}_2$, where $q \leq 471$.
- $T = \mathrm{P}\Omega_9(q)$, T_v is a $\mathcal{C}_1$-subgroup of type $\mathrm{GO}_1(q) \perp \mathrm{GO}_8^+(q)$, where $q \leq 19$.