# On the order of vertex-primitive 2-arc-transitive digraph besides direct cycles

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#### s-arc-transitive digraph

A digraph  $\Gamma$  is a pair  $(V, \rightarrow)$  with a set V (of vertices) and an antisymmetric irreflexive binary relation  $\rightarrow$  on V. (Note that by definition, if  $u \rightarrow v$ , then  $v \not\rightarrow u$ .)

- For a nonnegative integer s, an s-arc of  $\Gamma$  is a sequence  $(v_0, v_1, ..., v_s)$  of vertices with  $v_i \to v_{i+1}$  for each i = 0, ..., s 1.  $(v_i \neq v_{i+2}$  by the definition.)
- Γ is said to be s-arc-transitive if its automorphism group Aut(Γ) acts transitively on the set of s-arcs.
- s-arc-transitive  $\Rightarrow$  (s-1)-arc-transitive.

#### s-arc-transitive digraph

Let  $\Gamma$  be a (G, s)-arc-transitive digraph, where  $s \geq 1$ . Then  $\Gamma$  is vertex-transitive. The in-neighbours  $\Gamma(v)^- = \{u \in V(\Gamma) | u \to v\}$  and out-neighbours  $\Gamma(v)^- = \{w \in V(\Gamma) | v \to w\}$  has same order, called the *valency* of  $\Gamma$ .

- If the valency is 1, then  $\Gamma$  is a directed cycle, and s can be as large as possible.
- If the valency is 2 and G acts primitive on the vertex set  $V(\Gamma)$  of  $\Gamma$ , then  $\Gamma$  is a (undirected) cycle of prime order. (So vertex-primitive 2-arc-transitive digraphs besides direct cycles have valency at least 3.)
- Praeger [1] constructed a family of connected s-arc-transitive digraph  $C_r(v, s)$  of order  $rv^s$  and valency v, where  $r \ge 3, v \ge 2, s \ge 1$ . (s can be as large as possible)

<sup>&</sup>lt;sup>1</sup>Praeger, C. E. (1989) Highly Arc Transitive Digraphs. European Journal of Combinatorics, 10, 281-292.

#### Vertex-primitive s-arc-transitive digraph

- (1) Vertex-primitive s-arc-transitive digraph was first studied by Praeger [1] in 1989. She studied vertex-primitive s-arc-transitive digraph in each O'Nan-Scott type. It was proved that a (G, s)-arc-transitive digraph is a directed cycle if G contains a regular normal subgroup, where  $s \ge 2$ . (So if G is vertex-primitive of type HA, HS, HC, TW, then the digraph is directed cycle.)
- (2) Giudici and Xia studied the vertex-quasiprimitive (G, s)-arc-transitive digraph such that G is of type AS, CD, SD or PA).
  - (a) They characterized those digraphs from types SD and proved those digraphs are not 3-arc-transitive. (In particular, the order of graph is  $|T|^{|T|-1}$ .)
  - (b) They proved digraphs of type CD are direct product of digraphs of type SD and those digraphs are not 3-arc-transitive.
  - (c) They proved digraphs of type PA are direct product of digraphs of type AS and those digraphs is s-arc-transitive.
  - (d) They prompted a quastion:

Question: Is there an upper bound on s for vertex-primitive s-arc-transitive digraphs that are not directed cycles?

<sup>&</sup>lt;sup>1</sup>Praeger, C. E. (1989) Highly Arc Transitive Digraphs. European Journal of Combinatorics, 10, 281-292.

<sup>&</sup>lt;sup>2</sup>Giudici, M. & B. Xia (2017) Vertex-quasiprimitive 2-arc-transitive digraphs. Ars Mathematica Contemporanea, 14, 67-82.

- (3) Giudici,Li and Xia [3] studied the vertex-primitive (G, s)-arc-transitive digraph where G is almost simple with socle  $PSL_n(q)$ . Then proved  $s \leq 2$ , and provided many techniques.
- (4) Pan, Wu and Yin studied the case G is almost simple with socle  $A_n$ . They proved  $s \leq 2$  if the stabilizer is not a wreath product group. (The unsolved case was solved by Chen, Li and Xia[5] recently.)
- (5) Chen, Giudici and Praeger [6] studied the case G is almost simple with socle  ${}^{2}B_{2}(q)$  or  ${}^{2}G_{2}(q)$ . They proved that  $s \leq 1$ .

 $^3{\rm M.}$  Giudici, C. H. Li & B. Xia (2017) An infinite family of vertex-primitive 2-arc-transitive digraphs. Journal of Combinatorial Theory, Series B, 127, 1-13.

<sup>&</sup>lt;sup>4</sup>J. Pan, C. Wu & F. Yin (2020) Vertex-primitive s-arc-transitive digraphs of alternating and symmetric groups. Journal of Algebra, 544, 75-91.

 $<sup>^5</sup>$ J. Chen, J. Li & B. Xia (2021) Bounding s for vertex-primitive 2-arc-transitive digraph of alternating and symmetric groups. arXiv:2111.06579v1

<sup>&</sup>lt;sup>6</sup>L. Chen, M. Giudici, &C.E. Praeger (2021) Vertex-primitive s-arc-transitive digraphs admitting a Suzuki or Ree group. arXiv:2109.10508v1

(6) Giudici, Li and Xia [7] constructed an infinite family of vertex-primitive 2-arc-transitive digraphs that are not directed cycles (the first examples). The graphs are (PSL<sub>3</sub>(p<sup>2</sup>), 2)-arc-transitive with stabilizer A<sub>6</sub>, where p ≡ ±2(mod 5) and p ≥ 7.

So the example is of smallest order is of order  $|PSL_3(7^2)|/|A_6| = 30758154560$ .

Question: Whether exists vertex-primitive 2-arc-transitive digraph besides directed cycles with order less than 30758154560?

(This question was introduced by Xia in the summer of 2019 when visiting our university.)

<sup>&</sup>lt;sup>7</sup>M. Giudici, C.H. Li & B. Xia (2017) An infinite family of vertex-primitive 2-arc-transitive digraphs. Journal of Combinatorial Theory, Series B, 127, 1-13.

## Properties of 2-arc-transitive digraphs

# Proposition 1 ([GLX2019,Lemma 2.13])

For any vertex-primitive arc-transitive digraph  $\Gamma$ , either  $\Gamma$  is a directed cycle of prime length or  $\Gamma$  has valency at least 3.

#### Proposition 2 ([GLX2019,Lemma 2.14])

Let  $\Gamma$  be a connected G-arc-transitive digraph with arc (v, w). Let  $g \in G$  such that  $v^g = w$ . Then each nontrivial normal subgroup of  $G_v$  is not normalized by g.

### Proposition 3 ([GX2017,Lemma 2.2])

Let  $\Gamma$  be a *G*-arc-transitive digraph with a 2-arc (u, v, w). Then  $\Gamma$  is (G, 2)-arc-transitive if and only if  $G_v = G_{uv}G_{vw}$ . (homogeneous factorization)

# Proposition 4 ([GX2017,Corrollary 2.11])

Let  $\Gamma$  be a (G, 2)-arc-transitive digraph, M be a vertex-transitive normal subgroup of G. Then  $\Gamma$  is M-arc-transitive.

Let G be a transitive permutation group acting on  $\Omega$  and let  $v \in \Omega$ . Then a  $G_v$ -orbit on  $\Omega$  are called G-suborbit relative to v. For a G-suborbit  $w^{G_v}$ , if  $(v, w)^G = (w, v)^G$ , then  $w^{G_v}$  is said to be self-paired, otherwise non-self-paired.

Note that a *G*-arc-transitive digraph arises from a non-self-paired *G*-suborbit, equivalently, arises from a  $(G_v, G_v)$ -double coset  $G_v g G_v$  with  $g^{-1} \notin G_v g G_v$ .

#### Lemma 1

Let  $\Gamma$  be a (G, 2)-arc-transitive digraph with arc (v, w). Let  $g \in G$  such that  $v^g = w$ . Then w is in a non-self-paired G-suborbit and  $g^{-1} \notin G_v g G_v$ . Let  $G = PSL_3(3).2$ ,  $H = A\Gamma L_1(9)$ . Find representations of (H, H)-double cosets in G, and check whether  $g^{-1} \in HgH$  for each representation g.

```
G:=AutomorphismGroupSimpleGroup("L",3,3);
Gas:=Subgroups(G:OrderEqual:=9*8*2);
Ga:=Gas[1] 'subgroup;
gs,_:=DoubleCosetRepresentatives(G, Ga, Ga);
g:=gs[1];
Gab:=Ga meet Ga^g;
Tr:=Transversal(Ga.Gab);
Tr1:=[t:t in Tr| g*t*g in Ga];
if #Tr1 ge 1 then
print "the suborbit is self-paired";
else
print "the suborbit is non-self-paired";
end if;
```

#### The almost simple case

From now on, we consider the case G is almost simple.

#### Hypothesis

Let  $\Gamma$  be a connected *G*-vertex-primitive (G, 2)-arc-transitive digraph of valency at least 3, where *G* is almost simple with socle *T*. Take an arc  $u \to v$  of  $\Gamma$ . Let *g* be an element of *G* such that  $u^g = v$  and let  $w = v^g$ . Then  $u \to v \to w$  is a 2-arc in  $\Gamma$ .

We have seen that:

- $|G_v:G_{vw}| \geq 3$ , and
- $g^{-1} \notin G_v g G_v$ , or equivalently,  $\Gamma$  arises from a non-self-paired G-suborbit.
- $\Gamma$  is *T*-arc-transitive, and so  $\Gamma$  also arises from a non-self-paired *T*-suborbit.
- $G_v = G_{uv}G_{vw}$  is a homogeneous factorization and  $G_{uv}^g = G_{vw}$ . (In particular,  $G_{uv}$  is not conjugate to  $G_{vw}$  in  $G_v$ .)

#### The arc-transitivity of T

#### Why consider the arc-transitivity of T?

Note that  $T \leq G \leq \operatorname{Aut}(T)$ . There may be many candidates for G. If T has good property, then we only need to do computation in T, not in all candidates for G.

#### Lemma 2

 $\Gamma$  is *T*-arc-transitive. Let *t* be the orbits of  $T_{uv}$  on  $\Gamma(v)^+$  and o = |G|/|T|.

(1)  $t = |T_v|/|T_{uv}T_{vw}|$  and t divides o;

(2) If  $T_v = T_{uv}T_{vw}$ , then  $\Gamma$  is (T, 2)-arc-transitive;

(3) If  $|T_v| \ge o$ , then  $T_{uv}$  is not conjugate to  $T_{vw}$  in  $T_v$ .

(Note that  $T_{uv}$  is a normal subgroup of  $G_{uv}$  and  $G_{uv}$  is transitive on  $\Gamma(v)^+$ . Consider the action of  $T_{uv}$  on  $\Gamma(v)^+$ .)

#### nonsolvable composition factor of $G_v$

Let  $\mathbf{R}(G_v)$  be the largest solvable normal subgroup of  $G_v$ . Set

$$\overline{G_v} = G_v / \mathbf{R}(G_v), \ \overline{G_{uv}} = G_{uv} \mathbf{R}(G_v) / \mathbf{R}(G_v), \ \text{and} \ \overline{G_{vw}} = G_{vw} \mathbf{R}(G_v) / \mathbf{R}(G_v).$$

Then

- From the factorization  $G_v = G_{uv}G_{vw}$ , we have  $\overline{G_v} = \overline{G_{uv}G_{vw}}$ .
- $\overline{G_{uv}}$  and  $\overline{G_{vw}}$  has the same nonsolvable composition factors (count with multiplicities). (In particular, both solvable or both nonsolvable).
- If  $G_v$  has only one nonsolvable composition factor, then  $\overline{G_v}$  is almost simple.

#### Factorization of almost simple groups: two factors are solvable

For an integer n, we use  $\pi(n)$  be the set of primes dividing n; for a group G, we use  $\pi(G)$  be the set of primes dividing the order of G. For convenience, we define two sets of simple groups

$$\mathcal{T}_1 := \{ A_6, M_{12}, Sp_4(q)(q \text{ even}), P\Omega_8^+(q) \}.$$
  
$$\mathcal{T}_2 := \{ PSL_2(q), PSL_3(3), PSL_3(4), PSL_3(8), PSU_3(8), PSU_4(2) \}.$$
 (1)

#### Proposition 6

Let H be an almost simple group with socle M. Suppose M = KL, where K and L are solvable and  $M \not\leq K, L$ , then (H, K, L) are determined by [8,Proposition 4.1], in particular,  $M \in \mathcal{T}_2$  and  $\pi(K) \neq \pi(L)$ .

<sup>&</sup>lt;sup>8</sup>C.H. Li, B.Z. Xia, Factorizations of almost simple groups with a solvable factor, and Cayley graphs of solvable groups, arXiv :1408.0350.

PROPOSITION 4.1. Let G be an almost simple group with socle L. If G = HKfor solvable subgroups H, K of G, then interchanging H and K if necessary, one of the following holds.

- (a)  $L = \text{PSL}_2(q), H \cap L \leq D_{2(q+1)/d}$  and  $q \leq K \cap L \leq q:((q-1)/d)$ , where q is a prime power and d = (2, q-1).
- (b) L is one of the groups: PSL<sub>2</sub>(7) ≅ PSL<sub>3</sub>(2), PSL<sub>2</sub>(11), PSL<sub>3</sub>(3), PSL<sub>3</sub>(4), PSL<sub>3</sub>(8), PSU<sub>3</sub>(8), PSU<sub>4</sub>(2) ≅ PSp<sub>4</sub>(3) and M<sub>11</sub>; moreover, (G, H, K) lies in Table 4.1

Conversely, for each prime power q there exists a factorization G = HK satisfying part (a) with  $soc(G) = L = PSL_2(q)$ , and each triple (G, H, K) in Table 4.1 gives a factorization G = HK.

$\operatorname{row}$	G	H	K
1	$PSL_2(7).O$	7: $\mathcal{O}$ , 7: $(3 \times \mathcal{O})$	$S_4$
2	$PSL_2(11).O$	$11:(5 \times O_1)$	$A_4.O_2$
3	$PSL_2(23).O$	$23:(11 \times O)$	$S_4$
4	$PSL_3(3).O$	$13:O, 13:(3 \times O)$	$3^2:2.S_4$
5	$PSL_3(3).O$	$13:(3 \times O)$	$A\Gamma L_1(9)$
6	$PSL_3(4).(S_3 \times O)$	$7:(3 \times O).S_3$	$2^4:(3 \times D_{10}).2$
7	$PSL_3(8).(3 \times O)$	$73:(9 \times O_1)$	$2^{3+6}:7^2:(3 \times \mathcal{O}_2)$
8	$PSU_3(8).3^2.O$	$57:9.O_1$	$2^{3+6}$ :(63:3). $O_2$
9	$PSU_4(2).O$	$2^4:5$	$3^{1+2}_{+}:2.(A_4.\mathcal{O})$
10	$PSU_4(2).O$	$2^4:D_{10}.O_1$	$3^{1+2}_+:2.(A_4.\mathcal{O}_2)$
11	$PSU_4(2).2$	$2^4:5:4$	$3^{1+2}_+:S_3, 3^3:(S_3 \times \mathcal{O}),$
			$3^3:(A_4 \times 2), 3^3:(S_4 \times \mathcal{O})$
12	$M_{11}$	11:5	M <sub>9</sub> .2

TABLE.	4.1	L
		м.,

where  $\mathcal{O} \leq C_2$ , and  $\mathcal{O}_1, \mathcal{O}_2$  are subgroups of  $\mathcal{O}$  such that  $\mathcal{O} = \mathcal{O}_1 \mathcal{O}_2$ .

# Factorization of almost simple groups: two factors have same nonsolvable composition factor

#### Lemma 3

Let H be an almost simple group with socle M. Suppose H = KL with nonsolvable core-free subgroups K and L such that K and L have the same nonsolvable composition factors and the same multiplicities. Then  $H = (K \cap M)(L \cap M)$  with  $M \in \mathcal{T}_1$ , and interchanging K and L if necessary, one of the following holds:

(1) 
$$M = A_6, (H, K, L) \cong (A_6, A_5, A_5)$$
 or  $(S_6, S_5, S_5)$ .

(2) 
$$M = M_{12}, (H, K, L) \cong (M_{12}, M_{11}, M_{11}).$$

(3) 
$$M = \operatorname{Sp}_4(q)$$
 with  $q \ge 4$  even,  $H \le \operatorname{P}\Gamma\operatorname{Sp}_4(q)$ , and  
 $(K \cap M, L \cap M) \cong (\operatorname{Sp}_2(q^2).2, \operatorname{Sp}_2(q^2).2)$  or  $(\operatorname{Sp}_2(q^2).2, \operatorname{Sp}_2(q^2))$ 

(4)  $M = P\Omega_8^+(q), H \le P\Gamma O_8^+(q), \text{ and } (K \cap M, L \cap M) \cong (P\Omega_7(q), P\Omega_7(q)).$ 

( It is easy to prove by using the result of [9-11].)

<sup>&</sup>lt;sup>9</sup>C.H. Li, B. Xia, Factorizations of almost simple groups with a factor having many nonsolvable composition factors. Journal of Algebra 528 (2019) 439-473.

 $<sup>^{10}\</sup>mathrm{C.H.}$  Li, L. Wang, B. Xia, The exact factorizations of almost simple groups, arxiv 2012.09551v2.

<sup>&</sup>lt;sup>11</sup>R.W. Baddeley, C.E. Praeger, On classifying all full factorisations and multiple-factorisations of the finite almost simple groups. Journal of Algebra 204 (1998) 129-187.

#### The case $G_v$ has only one nonsolvable composition factor

For a group X, let  $X^{(\infty)}$  be the smallest normal subgroup of X such that  $X/X^{(\infty)}$  is soluble.

A group X is called quasisimple if X = X' and  $X/\mathbb{Z}(X)$  is simple. Note that X is the unique subgroup of X which has a composition factor isomorphic to  $X/\mathbb{Z}(X)$ . (If Y < X has a composition factor isomorphic to  $X/\mathbb{Z}(X)$ , then  $Y' = (Y\mathbb{Z}(X))' = X' = X$ , a contradiction.)

> $\mathcal{T}_1 := \{ A_6, M_{12}, Sp_4(q)(q \text{ even}), P\Omega_8^+(q) \}.$  $\mathcal{T}_2 := \{ PSL_2(q), PSL_3(3), PSL_3(4), PSL_3(8), PSU_3(8), PSU_4(2) \}.$

#### Lemma 4

Suppose that  $G_v$  has only one nonsolvable composition factor M.

- (1) If  $G_v$  is almost simple, then  $\Gamma$  is (T, 2)-arc-transitive, and  $(G_v, G_{uv}, G_{vw})$  satisfies (1)-(4) of Lemma 3.
- (2) If M ∉ T<sub>1</sub> ∪ T<sub>2</sub>, then both G<sub>uv</sub> and G<sub>vw</sub> have a nonsolvable composition factor M and G<sub>v</sub><sup>(∞)</sup> is not quasisimple.
- (3) If  $G_v^{(\infty)}$  is quasisimple, then  $M \in \mathcal{T}_1 \cup \mathcal{T}_2$ , and the factorization  $\overline{G_v} = \overline{G_{uv}G_{vw}}$  satisfies Lemma 3 or Proposition 6.

(1) was proved in [GLX2019,Corollary 3.4]. (2) and (3) was proved similarly. The key is Proposition 2, that is, each nontrivial normal subgroup of  $G_v$  is not normalized

hv a Fu-Gang Yin (BJTU) I use an examples to illustrate the proof of (2). Let  $G = Co_1$ ,  $G_v = 3.Suz$ :2.

- Then  $G_v^{(\infty)} = 3.Suz$  and  $\overline{G_v} = Suz$ :2.
- Since  $Suz \notin \mathcal{T}_1 \cap \mathcal{T}_2$ , both  $\overline{G_{uv}}$  and  $\overline{G_{vw}}$  contains a composition factor Suz, so do  $G_{uv}$  and  $G_{vw}$ .
- Note that  $G_{uv}^{(\infty)} \leq G_{uv} \cap G_v^{(\infty)} \leq G_v^{(\infty)}$  as  $G_{uv}/(G_{uv} \cap G_v^{(\infty)})$  is solvable and  $G_{uv}^{(\infty)}$  is the smallest normal subgroup N of  $G_{uv}$  such that  $G_{uv}/N$  is solvable.
- Since  $G_v^{(\infty)}$  is quasisimple,  $G_{uv}^{(\infty)} = G_v^{(\infty)}$ . Similarly,  $G_{vw}^{(\infty)} = G_v^{(\infty)}$ .
- Let  $Y = (G_{uv}^{(\infty)})^g$ . Then  $Y/(Y \cap G_{vw}^{(\infty)}) \cong YG_{vw}^{(\infty)}/G_{vw}^{(\infty)} \leq G_{vw}/G_{vw}^{(\infty)}$  is solvable. Thus  $Y \cap G_{vw}^{(\infty)}$  has a composition factor *Suz*.
- Since  $G_{vw}^{(\infty)}$  is quasisimple,  $Y \cap G_{vw}^{(\infty)} = G_{vw}^{(\infty)}$  and so  $Y = G_{vw}^{(\infty)}$ , that is  $(G_{uv}^{(\infty)})^g = G_{vw}^{(\infty)} = G_v^{(\infty)}$ , which contradicts Proposition 2.

#### **Computation method**

The homogeneous factorization  $G_v = G_{uv}G_{vw}$  implies  $G_{uv}$  and  $G_{vw}$  are not conjugate in  $G_v$  but in G, and

$$|G_v||G_{vu} \cap G_{vw}| = |G_{uv}||G_{vw}| = |G_{uv}|^2.$$

- We use Magma commands AutomorphismGroupSimpleGroup and MaximalSubgroups to construct G and G<sub>v</sub>.
- Let  $|G_v| = p_1^{s_1} \dots p_t^{s_t}$ . Then  $|G_{vw}|$  is multiple of  $p_1^{\lfloor \frac{s_1}{2} \rfloor} \dots p_t^{\lfloor \frac{s_t}{2} \rfloor}$ . We use command **Subgroups(Gv :OrderMultipleOf:=m)** can compute all possibilities of  $G_{uv}$  and  $G_{vw}$ . Then check whether  $|G_v||G_{vu} \cap G_{vw}| = |G_{uv}||G_{vw}|$ .
- The step finding all candidates of  $G_{uv}$  and  $G_{vw}$  can be optimized when  $G_v$  has only one nonsolvable composition factors, or when  $G_v$  has a normal *p*-subgroup N such that  $G_v/N$  is small.

For example, Let G = Th.

- (1) Let  $G_v := 2^5 . \text{PSL}_5(2)$ . Consider the factorization  $\overline{G_v} = \overline{G_{uv}G_{vw}}$ , we have both  $G_{uv}$  and  $G_{vw}$  have a composition factor  $\text{PSL}_5(2)$ . Note that  $|\text{PSL}_5(2)| = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$ . So  $|G_{uv}| = |G_{vw}|$  is multiple of  $2^{13} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$ .
- (2) Let  $G_v := 3.[3^8].2S_4$ . Now G has a normal p-subgroup  $N = 3.[3^8]$  such that  $G_v/N \cong 2S_4$ . Note that  $G_v/N \cong (G_{uv}N/N)(G_{vw}N/N)$ . Since  $G_{uv} \cong G_{vw}$  and N is a 3-group, a Sylow 2-subgroup of  $G_{uv}N/N$  is isomorphic to  $G_{vw}N/N$ 's. By computing all factorizations of  $G_v/N$  satisfying that two factors have isomorphic Sylow 2-subgroups, we obtain  $G_{uv}N/N = G_{vw}N/N = G_v/N \cong 2S_4$ . Then  $|G_{uv}| = |G_{vw}|$  is multiple of  $2^4 \cdot 3^5$ .

- When T is a simple classical group of Lie type of large order, we may do computation only in T. The command **ClassicalMaximals** is used to constructed the preimage of  $T_v$  in the quasisimple group relative to T.
- However, the above MAGMA computational method is not feasible if |G| and  $|G:G_v|$  are very large, for example, T is a simple group of Lie type and  $T_v$  is a parabolic subgroup. This case is dealt with by considering the suborbits of T.

#### $T = A_n$

By the classification of maximal subgroups of alternating and symmetric groups,  $G_v$  satisfies one of the following:

- (1)  $G_v = (S_m \times S_k) \cap G$ , where n = m + k, m > k (intransitive case);
- (2)  $G_v = (S_m \wr S_k) \cap G$ , where n = mk, m > 1 and k > 1 (imprimitive case);
- (3)  $G_v = \operatorname{AGL}(k, p) \cap G$ , where  $n = p^k$  (affine case);
- (4)  $G_v = (S^k : (\text{Out}(S) \times S_k)) \cap G$  where S is a nonabellain simple group,  $k \ge 2$  and  $n = |S|^{k-1}$  (diagonal case);
- (5)  $G_v = (S_m \times S_k) \cap G$ , where  $n = m^k$ ,  $m \ge 5$  and  $k \ge 2$  (primitive wreath case);
- (6)  $S \leq G_v \leq \operatorname{Aut}(S)$ , where S is a nonabelian simple group,  $S \neq A_n$  and  $G_v$  acts primitively on  $\{1, 2, ..., n\}$  (almost simple case).

The case (1) and (3) are impossible by PWY2020. If (4) or (5) happens, then  $|V(\Gamma)| > 30758154560$ .

#### almost simple case

Let  $\mathcal{N} := 30758154560$ . Suppose  $G_v$  is of almost simple case, that is,  $G_v \neq A_n$  is almost simple and H acts primitively on  $\{1, 2, ..., n\}$ .

- Now Soc $(G_v) \in \{A_6, M_{12}, PSp_4(2^f), P\Omega_8^+(q)\}.$
- From the upper bound for the order of primitive group given in [12,Theorem 1.1], we have  $|G_v| < n^{1+\lfloor \log_2(n) \rfloor}$  (with some exceptions). Then  $\frac{n!}{n^{1+\lfloor \log_2(n) \rfloor}} \leq \mathcal{N}$  implies  $n \leq 20$ .
- $PSp_4(2^f), P\Omega_8^+(q)$  has no primitive permutation representation of degree less than 20.
- If  $G_v = A_6$ , then n = 6 or 15 and  $G_{uv} \cong G_{vw} \cong A_5$ . When n = 6,  $G_{uv}$  is not conjugate to  $G_{vw}$  in G because one is transitive on  $\{1, ..., 6\}$  while the other is not. When n = 15,  $G_{uv}$  is not conjugate to  $G_{vw}$  in G by Magma.
- If  $G_v = M_{12}$ , then n = 12 and  $G_{uv} \cong G_{vw} \cong M_{11}$ . K and L are not conjugate in T.

 $<sup>^{12}\</sup>mathrm{A}.$  Maróti, On the orders of primitive groups. Journal of Algebra 258 (2002) 631-640.

#### imprimitive case

Suppose  $G_v$  is of imprimitive case, that is,  $G_v = (S_m \wr S_k) \cap G$ , where n = mk, m > 1and k > 1. Then  $|V(\Gamma)| = |S_n|/|S_m \wr S_k| = (mk)!/(m!)^k k!$ .

• By computation with computer, the (m, k) such that  $V(\Gamma) \leq \mathcal{N}$  are

$$\begin{aligned} (3 \le m \le 19, 2), (2 \le m \le 9, 3), (2 \le m \le 5, 4), \\ (2 \le m \le 4, 5), (3, 6 \le k \le 7), (2, 6 \le k \le 11). \end{aligned}$$

- The case  $(2 \le m \le 4, 5)$ ,  $(3, 6 \le k \le 7)$  and  $(2, 6 \le k \le 11)$  can be direct ruled out by computation in MAGMA.
- The case k = 2 with  $m \ge 11$ , and the case k = 3 and  $m \ge 7$  is difficult. (The reason is  $G_v$  is solvable, leading that  $G_v$  has many subgroups of order multiple of  $p_1^{\lfloor \frac{s_1}{2} \rfloor} .. p_t^{\lfloor \frac{s_t}{2} \rfloor}$ .) So we consider the suborbits of T.

#### suborbits of G with $G_v$ an imprimitive wreath product group

For two elements v, w in  $\Omega$ , where  $v = \{V_1, V_2, ..., V_k\}$  and  $w = \{W_1, W_2, ..., W_k\}$ , we have

$$V_i = \bigcup_{1 \le j \le k} (V_i \cap W_j), \text{ and } \sum_{1 \le j \le k} |V_i \cap W_j| = m.$$
$$W_j = \bigcup_{1 \le i \le k} (V_i \cap W_j), \text{ and } \sum_{1 \le i \le k} |V_i \cap W_j| = m.$$

We say the matrix  $M(v, w) := [|V_i \cap W_j|]_{k \times k}$  is a representation of the intersection of v and w. Note that the intersection of v and w may have may representations (if changing the order of  $V_i$  and  $W_j$  we may obtain a different representation), but all representations form the next set

 $\{P_1M(v, w)P_2|P_1, P_2 \text{ are } k \times k \text{ permutation matrix}\}.$ 

(Recall a  $k \times k$  permutation matrix is a matrix obtained by permuting the rows of an  $k \times k$  identity matrix according to some permutation on  $\{1, 2, ..., k\}$ .)

#### Lemma 5

Let  $G = S_n$ ,  $T = A_n$ ,  $\Omega$  the set of imprimitivity partitions of  $\{1, 2, ..., n\}$  with k blocks of size  $m, v, w \in \Omega$ , and let M(v, w) be a representation of the intersection of v and w. Then

- (1)  $w' \in w^{G_v}$  if and only if  $M(v, w') = P_1 M(v, w) P_2$ , where  $P_1, P_2$  are two  $k \times k$  permutation matrices.
- (2) v, w are interchanged by  $g \in G$  if and only if  $M(v, w)^{\mathrm{T}} = P_1 M(v, w) P_2$ , where  $M(v, w)^{\mathrm{T}}$  is the transpose of M(v, w) and  $P_1, P_2$  are two  $k \times k$  permutation matrices.
- (3) Suppose that v, w are interchanged by an odd permutation  $g \in G$ . Then v, w are interchanged by some  $t \in T$  if and only if  $G_{vw}$  contains odd permutations.

#### imprimitive case

Suppose  $G_v$  is of imprimitive case, that is,  $G_v = (S_m \wr S_k) \cap G$ , where n = mk, m > 1and k > 1. Then  $|V(\Gamma)| = |S_n|/|S_m \wr S_k| = (mk)!/(m!)^k k!$ .

• By computation, we remain the case k=2 with  $m\geq 11$ , and the case k=3 and  $m\geq 7$ 

Apply Lemma 5.

- If k = 2, then  $M_{vw}$  is always symmetric and hence all suborbits of T are self-paired.
- If k = 3, then  $2 \le m \le 9$ . By computation, the non-self-paired orbital cases are:

(1) 
$$m = 6$$
, and  $M_{vw} = \begin{bmatrix} 0 & 2 & 4 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$  or its transpose;  
(2)  $m = 7$ , and  $M_{vw} = \begin{bmatrix} 0 & 2 & 5 \\ 3 & 3 & 1 \\ 4 & 2 & 1 \end{bmatrix}$  or its transpose;  
(3)  $m = 8$ , and  $M_{vw} = \begin{bmatrix} 0 & 3 & 5 \\ 4 & 2 & 2 \\ 4 & 3 & 1 \end{bmatrix}$  or its transpose.

For the case m = 6, and  $M_{vw} = \begin{bmatrix} 0 & 2 & 4 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ :

- Now, n = 18,  $G_v = S_6 \wr S_3 \cap G$ ,  $G_{vw} = (S_3^2 \times S_2^3 \times S_4):S_2 \cap G$ .
- Let  $N = S_m^k \cap G$  be the base group of  $G_v$ , then  $N \trianglelefteq G_v$  and  $G_v/N \cong S_n$  (This is clear when  $G = S_n$ . If  $G = A_n$ , then N is with index 2 in  $S_m^k$  as  $S_m^k$  contains odd permutations, so  $|G_v/N| = |S_k|$  and hence  $G_v/N \cong S_k$ ).
- In the factorization  $S_k \cong G_v/N = (G_{uv}N/N)(G_{vw}N/N)$ , one factor should be transitive (on k points) (see [13,1.3,1.4] for a proof).
- Now  $G_{vw}N/N \cong S_2$  and  $G_{uv}N/N \cong 1$ , a contradiction. (A subgroup of  $G_v/N = S_3$  describes the symmetry of rows. In  $M_{vw}$ , the second and third row can be interchanged, so  $G_{vw}N/N \cong S_2$ . While for  $G_{uv}$ , we have
  - $M_{vu} = \begin{bmatrix} 0 & 3 & 3 \\ 2 & 2 & 2 \\ 4 & 1 & 1 \end{bmatrix}, \text{ there is no pair of rows can be interchanged, this means } G_{uv} \leq N \text{ and } G_{uv}N/N \cong 1. \text{ })$

<sup>&</sup>lt;sup>13</sup>J. Wiegold and A.G. Williamson, The factorization of the alternating and symmetric groups, Math. Z. 175 (1980), 171-179.

#### T is a simple sporadic group

#### Lemma 6

Suppose that Hypothesis holds, then T is not one of the next 22 groups:

 $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, HS, J_2, McL, Suz, J_3, Co_3, Co_2, He, Fi_{22}, Ru.$ Th, Fi<sub>23</sub>, J<sub>4</sub>, Ly, HN, O'N.

- If T is one group in the first row, then G has a permutation representation of small degree (less than 10000). We can obtain G and  $G_v$  in MAGMA using its commands, then computation shows that  $G_v$  has no homogeneous factorization  $G_v = KL$  such that  $|G_v: K| \geq 3$  and K and L are conjugate in G.
- Let T be one group in the second row. If  $T = Fi_{23}$ , then we can construct G and  $G_v$  using the MAGMA commands. For other groups, we can construct G and  $G_v$  using the information of generators given in the Web Atlas [14].

<sup>&</sup>lt;sup>14</sup>R. A. Wilson, S. J. Nickerson, J. N. Bray et al., An Atlas of Group Representations, ver. 3, http://brauer.maths.qmul.ac.uk/Atlas/v3/.

Note that the information of generators for maximal subgroups  $2^{3+2+6}$ . $(3 \times PSL_3(2))$ ,  $3^4:2.(A_4 \times A_4).4$  and  $(A_6 \times A_6).D_8$  of HN is not given in the Web Atlas.

- The groups  $2^{3+2+6}.(3\times PSL_3(2))$  and  $3^4:2.(A_4\times A_4).4$  can be constructed using the method as in [15, p.318]. For example, to construct  $2^{3+2+6}.(3\times PSL_3(2))$ , we can first construct a Sylow 2-subgroup P of HN (note that  $|HN|_2 = |2^{3+2+6}.(3\times PSL_3(2))|_2 = 2^{14}$ ), then compute all normal 2-subgroups of order  $2^{11}$  of P, and their normalizer in HN; if the normalizer has order  $|2^{3+2+6}.(3\times PSL_3(2))|$ , then the normalizer is the required maximal group  $2^{3+2+6}.(3\times PSL_3(2))$ .
- The group  $(A_6 \times A_6).D_8$  has a normal subgroup  $(A_6 \times A_6).2^2$  contained in  $A_{12}$  (the information of generators of  $A_{12}$  is given in Web Atlas), so we can construct the group  $(A_6 \times A_6).2^2$  in  $A_{12}$  first and then compute its normalizer, which is the required  $(A_6 \times A_6).D_8$ .

<sup>&</sup>lt;sup>15</sup>T.C. Burness, E.A. O'Brien, R.A. Wilson, Base sizes for sporadic simple groups. Israel J. Math. 177 (2010) 307-333.

- If  $G_v$  has only one nonsolvable composition factor and  $G_v^{(\infty)}$  is quasisimple, then we may apply Lemma 4 to rule out this candidate directly. For example G = Thand  $G_v$  is  ${}^{3}D_4(2)$ :3, or PSU<sub>3</sub>(8):6, or PSL<sub>2</sub>(19):2, or PSL<sub>3</sub>(3), or  $M_{10}$  (note that  $M_{10}$  is not a subgroup of S<sub>6</sub>), or S<sub>5</sub>, or (3×G<sub>2</sub>(3)):2.
- If  $G_v$  is a metacyclic Frobenius group, then we may apply Proposition 2 to rule out it. For example, G = Th and  $G_v = 31:15$ . Now from the homogeneous factorization  $G_v = G_{uv}G_{vw}$  we see both  $G_{uv}$  and  $G_{vw}$  contains the normal subgroup  $M \cong C_{31}$  of  $G_v$ . Since  $G_v$  has only one subgroup isomorphic to  $C_{31}$ ,  $M^g = M$ , which contradicts Proposition 2.

There are two cases is difficult to compute.

•  $G = Fi_{23}$  and  $G_v = 3^{1+8} \cdot 2^{1+6} \cdot 3^{1+2} \cdot 2S_4$ . Then  $|G_{uv}|$  is multiple of  $2^6 \cdot 3^7$  as  $|G_v| = 2^{11} \cdot 3^{13}$ . It is difficult to compute all subgroups of order multiple of  $2^6 \cdot 3^7$  in  $G_v$ . So we take  $N = 3^{1+8}$  and consider the factorizations of  $2^{1+6} \cdot 3^{1+2} \cdot 2S_4 \cong G_v/N = (G_{uv}N/N)(G_{vw}N/N)$ , where  $G_{uv}N/N, G_{vw}N/N$  have order multiple of  $2^6$ . Since  $G_{uv} \cong G_{vw}, G_{uv}/O_3(G_{uv}) \cong G_{vw}/O_3(G_{vw})$ . Note that  $G_{uv} \cap N \leq O_3(G_{uv})$  and  $G_{vw} \cap N \leq O_3(G_{vw})$ . Thus  $G_{uv}N/N \cong G_{uv}/(G_{uv} \cap N)$  has a normal 3-subgroup  $M_1$  isomorphic  $O_3(G_{uv})/(G_{uv} \cap N)$ , and  $G_{vw}N/N \cong G_{vw}/(G_{vw} \cap N)$  has a normal 3-subgroup  $M_2$  isomorphic  $O_3(G_{vw})/(G_{vw} \cap N)$  such that  $(G_{uv}N/N)/M_1 \cong (G_{vw}N/N)/M_2$ . By computing all factorizations of  $2^{1+6} \cdot 3^{1+2} \cdot 2S_4$ , we find no desired factorization. Therefore this case is impossible.

• G = HN:2 and  $G_v = (S_6 \times S_6) \cdot 2^2$ . Computation shows that  $G_v$  indeed has homogeneous factorization  $G_v = KL$ , where  $K \cong L$  and  $|G_v: K| \ge 3$ . However, some computation evidences show that K and L is not conjugate in G. We do as follows. The groups G and  $G_v$  are constructed by  $133 \times 133$  matrices over  $\mathbb{F}_5$ (note that the minimal degree of permutation representation of G is 1140000 and it is too large for computation.) Computation shows that  $G_v$  has 10 homogeneous factorizations  $G_v = KL$ , 8 of them with |K| = 1440 and the other 2 with |L| = 2880. The difficulty is that the MAGMA command **IsConjugate** is not valid when checking whether K and L. So we compute the conjugacy classes of K and L to check whether K and L are conjugate in G. In a homogeneous factorization  $G_v = KL$ , we find there is an element of order 2 in L such that it is not similar to any element of order 2 in L (using the MAGMA command **IsSimilar**). This implies K and L is not conjugate in G as  $G \leq GL_{133}(5)$ . For other homogeneous factorizations, we also find such element. Therefore this case is impossible.

#### Lemma 7

Suppose that Hypothesis holds and suppose that  $|V\Gamma| \leq \mathcal{N}$ . Then T is not one of  $\mathbb{M}, \mathbb{B}, Fi'_{24}, Co_1$ .

- The Monster group  $\mathbb{M}$  has no maximal subgroup of index no more than  $\mathcal{N}$ .
- If  $T = \mathbb{B}$ , then G = T and  $G_v = 2.^2 E_6(2):2$ .
- If  $T = Fi'_{24}$ , then G = T and  $G_v = Fi_{23}$ ,  $2 \cdot Fi_{22}$ :2,  $(3 \times P\Omega_8^+(3):3)$ :2. (Note that if  $T_v = (3 \times P\Omega_8^+(3):3)$ :2, then  $\overline{G_v} = P\Omega_8^+(3)$ :S<sub>3</sub> contains a graph automorphism of order 3 and hence  $\overline{T_v}$  is not a subgroup of  $P\Gamma O_8^+(q)$ . This can be verified by MAGMA).
- If  $T = Co_1$ , then G = T and the possible  $G_v$  are

 $Co_{2}, 3.Suz.2, Co_{3}, \text{PSU}_{6}(2):\text{S}_{3}, (\text{A}_{4} \times G_{2}(4)):2,$  $2^{1+8}.\text{P}\Omega_{8}^{+}(2), 2^{11}:M_{24}, 2^{2+12}:(\text{A}_{8} \times \text{S}_{3}), 2^{4+12}.(\text{S}_{3} \times 3.\text{S}_{6}), 3^{2}.\text{PSU}_{4}(3).\text{D}_{8}, 3^{6}:2.M_{12}.$ 

#### T is a simple group of Lie type

Some helpful results:

- Alavi and Burness [16] determined all large maximal subgroups of finite simple groups and their automorphism groups. Formally, a subgroup X is called *large* in group Y if  $|X| > |Y|^{1/3}$ . Note that if  $|T|^{2/3} > \mathcal{N}$ , then our assumption  $|V(\Gamma)| = |T: T_v| \leq \mathcal{N}$  implies that  $T_v$  is large in T.
- Maximal parabolic subgroups are always subgroups of T with small index. So we need often considering the case that  $T_v$  is a parabolic subgroup. If  $T_v$  is a parabolic subgroup, then the information of T-suborbits can be computed by computing on the Weyl group of T and roots of T. See [17, Chapter 2].
- Degree of the minimal permutation representation of T, see [18, Table 4]).

<sup>&</sup>lt;sup>16</sup>S.H. Alavi, T.C. Burness, Large subgroups of simple groups. J. Algebra 421 (2015) 187-233.

<sup>&</sup>lt;sup>17</sup>R.W. Carter, Finite Groups of Lie Type: Conjugacy Classes and Complex Characters. John Wiley & Sons, New York, 1985.

<sup>&</sup>lt;sup>18</sup>S. Guest, J. Morris, C.E. Praeger, P. Spiga, On the maximum orders of elements of finite almost simple groups and primitive permutation groups. Transactions of the American Mathematical Society 367 (2015) 7665-7694.

#### Lemma 8

Suppose that T is a simple group of Lie type such that  $|T|^{2/3} \leq \mathcal{N}$  and  $T \neq \text{PSL}_2(q)$ . Then T is one of the following groups:

- (1)  $PSL_n(q)$ , or  $PSU_n(q)$ , where (n,q) is  $(3, q \le 97)$ ,  $(4, q \le 11)$ ,  $(5, q \le 4)$ , (6, 2), or (7, 2);
- (2)  $PSp_n(q)$ , where (n,q) is  $(4,q \le 37)$ ,  $(6,q \le 5)$ , or (8,2);
- (3)  $P\Omega_n^{\epsilon}(q)$ , where (n,q) is (7,3), (7,5), (8,2), (8,3), or (10,2).
- (4)  ${}^{2}B_{2}(2^{3}), {}^{2}B_{2}(2^{5}), {}^{2}B_{2}(2^{7}), {}^{2}B_{2}(2^{9}), {}^{2}F_{4}(2)', {}^{2}G_{2}(3^{3}), {}^{3}D_{4}(2), {}^{3}D_{4}(3), F_{4}(2), G_{2}(q)(q \le 11).$

#### Parabolic subgroup

Let T be a simple group of Lie type in characteristic p. Let U be a Sylow p-subgroup of T.

- The normalizer of U in T is called a *Borel subgroup* of T, and B = U:H, moreover, H is called the *Cartan subgroup*.  $(H \neq 1.)$
- A subgroup P of T is called a *Parabolic subgroup* of T if P contains a conjugate of B.

When P is a standard parabolic subgroup, the method of how to computing the subdegrees of T (acting on [T : P]) were introduced in [19].

<sup>&</sup>lt;sup>19</sup>A.E. Brouwer, A.M. Cohen, Computation of some parameters of Lie geometries, Mathematisch Centrum. ZW, afdeling zuivere wiskunde, 1983, pp. 21.

# (B, N)-pair

A pair of subgroups B, N of a group G is called a (B, N)-pair if the following axioms are satisfied:

- (1)  $G = \langle B, N \rangle$ .
- (2)  $H := B \cap N$  is normal in N.
- (3) The group W := N/H (Weyl group) is generated by a set of involutions  $w_i$ ,  $i \in I$ .
- (4) If  $n_i \in N$  maps to  $w_i$  under the natural homomorphism N to into W, and if  $n \in N$ , then
  - (i)  $Bn_iB.BnB \subseteq Bn_inB \cup BnB$ .
  - (ii)  $n_i B n_i \neq B$ .

Some important properties:

- $G = \bigcup_{w \in W} BwB$ .
- any element w in W is a product of some  $w_i$ , the length of w, saying l(w), is smallest integer q > 0 such that w is the product of a sequence of q elements.
- $Bn_iB.BnB = Bn_inB \cup BnB$  if  $l(n_i\overline{n}) = l(\overline{n}) 1$ ;  $Bn_iB.BnB = Bn_inB$  if  $l(n_i\overline{n}) = l(\overline{n}) + 1$ ;

- For each  $J \subseteq I$ ,  $P_J := BN_J B$  is a subgroup of G, where  $N_J = H \langle n_i : i \in J \rangle$ . If  $J = \emptyset$ , then  $B = P_J$ . The group  $P_J$  is called the standard parabolic subgroup (associated with J).
- For a double coset  $W_J v W_J$  in W, there is a unique element  $w \in W_J v W_J$  such that w is with shortest length (such w is called (J, J)-reduced), see [17]. Let  $D_J$  be the set of (J, J)-reduced elements in W.
- For a coset  $W_J v$  in W, there is a unique element  $u \in W_J v$  such that u is with shortest length (such u is called  $(J, \emptyset)$ -reduced). Let  $R_J$  be the set of  $(J, \emptyset)$ -reduced elements in W.
- $PwP = BW_J BwBW_J B = BW_J wW_J B$ , where  $w \in D_J$ . As a consequence, each  $(P_J, P_J)$ -double coset PwP in G corresponds to a  $(W_J, W_J)$ -double coset  $W_J wW_J$  in W, and it is a bijection.

#### Proposition 6 ([19, Proposition 2])

Let notations be as above. Then G (on  $[G:P_J]$ ) has same rank and number of non-self-paired suborbits as its Weyl group W (on  $[W:W_J]$ ). A suborbit of Gcorresponds to  $P_J w P_J$ , where  $w \in D_J$ , is selfpaired if and only if w is an involution, and its length is  $\sum_{v \in R_J \cap w W_J} q^{l(v)}$ , where l(v) is the length of v.

#### computation in Magma

For example, to compute the rank and suborbit of  $G = PSL_5(2)$  acting on  $[G : P_J]$ , where  $P_J = [2^7]:PSL_3(2)$ , the stabilizer in G of a (1,4)-flag, we can do as follows

```
W := CoxeterGroup(GrpFPCox,"A4");J:={2,3};
DJ,_:=Transversal(W, J, J);print "the rank:", #DJ;
W0,phi := CoxeterGroup(GrpPermCox, W);
DJ1:=[phi(DJ[i]): i in [2..#DJ]| Order(DJ[i]) ne 2];
print "number of nonself-paired suborbits:", #DJ1;
RJ:=Transversal(W, J);
WJ:= StandardParabolicSubgroup(W0, J);
for g in DJ do
RJ_wWJ:=[w: w in RJ|g eq TransversalElt(W0,WJ,phi(W),WJ)];
RJ_wWJ1:=[Length(W): w in RJ_wWJ];
RJ_wWJ1; //the set [l(v):v \in R_J \cap wW_J]
end for;
```

Let  $T_v = P_J$  be a parabolic subgroup of T. Now, we have find all non-self-paired T-suborbits and known their lengths.

- Since  $T_v g T_v = P_J g P_J$ , where  $g \in W$ ,  $T_{vw} = P_J \cap P_J^g$  and  $T_{uv} = P_J \cap P_J^{g^{-1}}$ .
- If we know  $|T_{uvw}| = |P_J \cap P_J^g \cap P_J^{g^{-1}}|$ , then we can check whether  $|T_v| = t|T_{uv}T_{vw}| = \frac{t|T_{uv}||T_{vw}|}{|T_{uvw}|}$ , where  $t \mid |G/T|$ .
- We can estimate the above equation by  $|T_v|_q |T_{uvw}|_q = t_q |T_{uv}|_q^2$ .
- The values of  $|T_v|_q$  and  $|T_{uv}|_q^2$  are known. The value of  $|T_{uvw}|_q$  can be estimated by computing on the roots of T.

# $|T_v|_q$ , $|T_{uv}|_q^2$ and $|T_{uvw}|_q$

The next results can be found or obtained from Carter's book.

- $P_J = \langle H, X_r | r \in \Phi^+ \cup \Phi_J \rangle$ , where  $\Phi_J$  is the set of roots spanned by fundamental roots in J. Moreover,  $|P_J|_q = q^{m_1}$ , where  $m_1$  is the number of positive roots in  $\Phi^+ \cup \Phi_J$ .
- $P_J \cap P_J^g = \langle H, X_r | r \in (\Phi^+ \cup \Phi_J) \cap (\Phi^+ \cup \Phi_J)^{g^{-1}} \rangle$ . Therefore,  $|P_J \cap P_J^g|_q = q^{m_2}$ , where  $m_2$  is the number of positive roots in  $(\Phi^+ \cup \Phi_J) \cap (\Phi^+ \cup \Phi_J)^{g^{-1}}$ .
- $B = \langle H, X_r | r \in \Phi^+ \rangle = UH$ . So  $B \cap B^g = \langle H, X_r | r \in \Phi^+ \cap (\Phi^+)^{g^{-1}} \rangle = (U \cap U^g)H$ .
- $B \cap B^g \cap B^{g^{-1}} = \langle H, X_r | r \in \Phi^+ \cap (\Phi^+)^{g^{-1}} \cap (\Phi^+)^g \rangle = (U \cap U^g \cap U^{g^{-1}})H.$ Therefore,  $|P_J \cap P_J^g \cap P_J^{g^{-1}}|_q$  is divisible by  $q^{m_3}$ , where  $m_3$  is the number of positive roots in  $\Phi^+ \cap (\Phi^+)^{g^{-1}} \cap (\Phi^+)^g.$

An example for  $T = G_2(q)$ 

Let  $T = G_2(q)$ ,  $T_v = [q^6]:(q-1)^2$  be a Borel subgroup of T, and G contains a graph automorphism of T.

- Computation on the Weyl group of T shows that T has rank 12 and there are 4 non-self-paired suborbits of length q<sup>2</sup> or q<sup>4</sup>.
  The Weyl group W of G<sub>2</sub>(q) is a dihedral group of order 12. So the rank is 12. In D<sub>12</sub>, there are 4 elements with order greater than 2. So there are 4 non-self-paired suborbits.)
- If the length is  $q^4$ , then  $|T_{vw}|_p^2 = q^4 < q^6 = |T_v|_p$ , a contradiction.
- Hence the length is  $q^2$  and so  $T_{vw} = [q^4]:(q-1)^2$ . By computing the roots,  $T_{uvw} = [q^2]:(q-1)^2$ . Then  $|T_v||T_{uvw}| = |T_{uv}||T_{vw}|$  and so  $T_v = T_{uv}T_{vw}$ , which means  $\Gamma$  is (T, 2)-arc-transitive. However, we next show that  $\Gamma$  is not (G, 2)-arc-transitive.

Let  $\Phi = \Phi^+ \cup \Phi^-$  be a root system of T with fundamental roots a, b, and  $\Phi^+ = \{b, a, b + a, b + 2a, b + 3a, 2b + 3a\}$ . Let H be the Cartan subgroup of T. Take  $T_v = \langle H, X_r : r \in \Phi^+ \rangle$  be the Borel subgroup. Label roots as follows:

b	a	b+a	b+2a	b+3a	2b+3a	-b	-a	-b-a	-b-2a	-b-a	-2b-3a
1	2	3	4	5	6	7	8	9	10	11	12

- $W = \langle w_a, w_b \rangle$ , where  $w_a = (1, 5)(2, 8)(3, 4)(7, 11)(9, 10)$  and  $w_b = (1, 7)(2, 3)(5, 6)(8, 9)(11, 12)$ . It can be view as a permutation group on  $\Phi$ .
- $G_v$  contains a graph automorphism  $\gamma$  normalizing  $T_v$ . By computation with MAGMA,  $\gamma = (1, 2)(3, 5)(4, 6)(7, 8)(9, 11)(10, 12)$ . (Graph automorphism can also be view as permutation group on  $\Phi$ , see [19, Section 12.4], and it can be contained from the normalizer of W in Sym( $\Phi$ ).)
- By computation, g = (1, 11, 12, 7, 5, 6)(2, 4, 3, 8, 10, 9) (or its inverse). Note that  $\Phi^+ \cap (\Phi^+)^{g^{-1}} = \{2, 4, 5, 6\}$  as  $1^{g^{-1}} = 6$ ,  $6^{g^{-1}} = 5$ ,  $3^{g^{-1}} = 4$  and  $4^{g^{-1}} = 2$ . So

$$T_{vw} := \langle H, X_r | r \in \{b, b+a, b+2a, 2b+3a\} \rangle.$$

<sup>&</sup>lt;sup>19</sup>R.W. Carter, Simple groups of Lie type, Wiley, London, 1972.

- Recall  $G_v$  contains a graph automorphism  $\gamma$  and  $|G_{vw}: T_{vw}| = |G_v: T_v| = |G: T|$ . So  $T_{vw}$  is normalized by some element  $h = \phi \gamma x \in G_{vw} \setminus T_{vw}$ , where  $x \in T_v = UH$  and  $\phi$  is a field automorphism.
- Let  $M = \langle X_r | r \in \{b, b+a, b+2a, 2b+3a\} \rangle$ . Then M is the largest normal p-subgroup of  $T_{vw}$ , and hence  $M = M^h = M^{\phi\gamma x} = M^{\gamma x}$  (as  $\phi$  normalizes each root subgroup)
- Now  $M^{\gamma} = \langle X_r | r \in \{a, b + 3a, b + 2a, 2b + 3a\} \rangle$  as  $\{1, 3, 4, 6\}^{\gamma} = \{2, 5, 4, 6\}.$
- Since *H* normalizes each root subgroup, *x* can be taken as an element in  $U = \langle X_r | r \in \Phi^+ \rangle$ . By the structure of *U*, *U* has a normal subgroup  $N := \langle X_{2b+3a}, X_{b+3a} \rangle$ .
- Since  $M = (M^{\gamma})^x$ ,  $M/N = (M^{\gamma}/N)^x$ . However,  $|M/N| = q^3$  while  $|M^{\gamma}/N| = q^2$ , a contradiction.

Difficult cases when  $T_v$  is a parabolic subgroup

- $T = G_2(q), T_v$  is a Borel subgroup.
- $T = PSp_4(q), T_v$  is a Borel subgroup.
- $T = P\Omega_8^+(q), T_v$  is of type  $A_1$ .

Difficult cases when  $T_v$  is not a parabolic subgroup

- $T = G_2(q),$ 
  - (i)  $T_v = \mathrm{SL}_2(q) \times \mathrm{SL}_2(q)$  with  $q \in \{4, 8, 16\}$ .
  - (ii)  $T_v = (\mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)).2$  with  $q \in \{3, 5, 7, 9, 11, 13, 17, 19\}.$
- $T = PSU_n(q), T_v$  is a  $\mathcal{C}_5$ -subgroup of type  $Sp_n(q)$ , where  $q \in \{4, 8, 16, 32, 64\}$ .
- $T = PSp_4(q), T_v$  is a  $C_2$ -subgroup of type  $Sp_2(q) \wr S_2$ , where  $q \le 471$ .
- $T = P\Omega_9(q), T_v$  is a  $C_1$ -subgroup of type  $GO_1(q) \perp GO_8^+(q)$ , where  $q \leq 19$ .