On Compatible Groups

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Throughout, all groups and digraphs are finite.
Let digraph Γ be G-arc-transitive and $v \in V\Gamma$.

$\Gamma^{-}(v)$: in-neighborhood

$\Gamma^{+}(v)$: out-neighborhood

$G_v^{\Gamma^{\epsilon}(v)}$: induced permutation group on $\Gamma^{\epsilon}(v)$, where $\epsilon \in \{+, -\}$.

Remark. $G_v^{\Gamma^{+}(v)}$ and $G_v^{\Gamma^{-}(v)}$ may not be isomorphic.

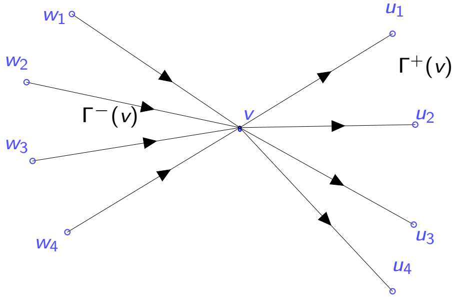

Figure: Local actions of G at v

Example

Coset digraph

Let $H \leq G$ and $e \notin S \subset G$. Define $\Gamma = Cos(G, H, HSH)$.

- $V\Gamma := [G : H]$.
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Let $G = C_4 \wr C_3 = \langle a \rangle \wr \langle g \rangle$ and $H := \langle a^2, a^g \rangle \cong C_2 \times C_4$. Consider Coset digraph $\Gamma = Cos(G, H, HgH)$. Let $v := H$.

- vertex stabilizer: $G_v = H$
- neighborhood: $\Gamma^+(v) = \{Hgh; h \in H\}$, $\Gamma^-(v) = \{Hg^{-1}h; h \in H\}$
- local actions: $G_v^{\Gamma^+(v)} \cong H/\operatorname{Core}_H(H \cap H^g) \cong C_2^2$,
  $G_v^{\Gamma^-(v)} \cong H/\operatorname{Core}_H(H \cap H^{g^{-1}}) \cong C_4$.

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The compatible groups

Two transitive groups L^+ and L^- are compatible if $L^+ \cong G_v^{\Gamma^+(v)}$ and $L^- \cong G_v^{\Gamma^-(v)}$ for some G-arc-transitive digraph Γ.

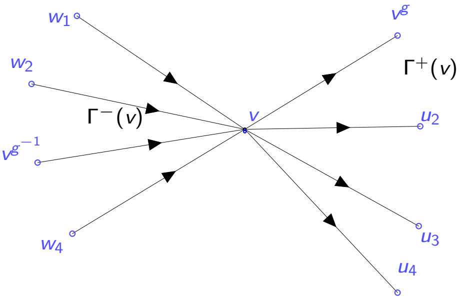
Problem

Given two permutation groups. Determine whether they are compatible.

Witness

Let $u_1 = v^g$. Then $v^{g^{-1}} \in \Gamma^-(v)$.

$G_v^{\Gamma^\epsilon(v)} \cong G_v^{[G_v : G_v \cap G_v^{g^\epsilon}]}$.



Fact

L^+ and L^- are compatible $\iff \exists\ G, H \leq G, g \in G$ s.t. $L^\epsilon \cong H^{[H:H \cap H^{g^\epsilon}]}$.

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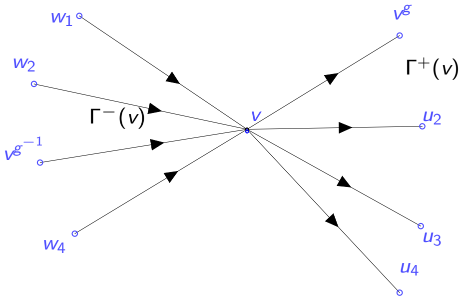

Fact

L^+ and L^- are compatible $\iff \exists\ G, H \leq G, g \in G$ s.t. $L^\epsilon \cong H^{[H:H \cap H^{g^\epsilon}]}$.

Question: Is G necessary?

Witness

Theorem (Giudici et al. 2019)

L^+ and L^- are compatible $\iff$ $\exists$ a group H with subgroups $K^+ \cong K^-$ s.t. $L^+ \cong H^{[H:K^+]}$ and $L^- \cong H^{[H:K^-]}$.

Let L^+ and L^- be transitive.

Witness: (G, H^+, H^-) with $H^+ \cong H^-$ such that $L^+ \cong G^{[G:H^+]}$ and $L^- \cong G^{[G:H^-]}$.

Remark. L^+ and L^- are compatible $\iff$ $\exists$ a witness of L^+ and L^-.

Theorem

Let (G, H^+, H^-) be a witness s.t. $|G|$ is minimal. Then $G \neq \mathrm{Core}_G(H^+)H^-$ or $G \neq \mathrm{Core}_G(H^-)H^+$.

Witness

The properties of "minimal witness" (G, H^+, H^-) are generally difficult to determine. But we can determine one of its quotients.

Theorem

Let L^+ and L^- be transitive and compatible with a "minimal witness" (G, H^+, H^-). Then $\exists\ N^\epsilon \lhd \neq L^\epsilon$ and isomorphism $\phi : L^+/N^+ \to L^-/N^-$ s.t.

$$G/(\mathrm{Core}_G(H^+) \cap \mathrm{Core}_G(H^-)) \cong \{(x, y) \in L^+ \times L^- \mid \phi(xN^+) = yN^-\}.$$

In particular, $G/(\mathrm{Core}_G(H^+) \cap \mathrm{Core}_G(H^-)) \cong (N^+ \times N^-).(L^+/N^+)$.

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Properties implying compatibility? None is known in general case.

Compatibillity of regular/abstract groups

Assume L^+ and L^- are regular.

Corollary (Giudici et al. 2019)

L^+ and L^- are compatible $\iff$ $\exists$ a group G with normal subgroups $H^+ \cong H^-$ s.t. $L^+ \cong G/H^+$ and $L^- \cong G/H^-$.

In particular, we can treat regular groups as abstract groups.

(Abstract) Compatible: Two (abstract) groups L^+ and L^- are compatible $\iff$ $\exists$ G with $N^+ \cong N^-$ s.t. $L^\epsilon \cong G/N^\epsilon$.

Compatibillity of regular/abstract groups

Let L^+ and L^- be two abstract groups. If $\exists$ two subnormal series

$$1 = N_0 \trianglelefteq \cdots \trianglelefteq N_n = L^+$$

and

$$1 = M_0 \trianglelefteq \cdots \trianglelefteq M_n = L^-$$

such that $N_{i+1}/N_i \cong M_{i+1}/M_i$ for all $0 \leq i \leq n-1$, then we say that L^+ and L^- have compatible subnormal series.

An easy induction yields the following theorem.

Theorem

Two compatible abstract groups have compatible subnormal series.

Compatibillity of regular/abstract groups

Weak forms of the converse can also be established.
Method: construct a witness.

Theorem

Let L^+ and L^- be abstract groups. If

- L^+ and L^- have compatible subnormal series of length 2 are compatible; or
- $|L^+| = |L^-| = pqr$, where p, q, r are distinct prime numbers; or
- L^+ and L^- are abelian and of the same order,

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Conjecture

(Abstract) L^+ and L^- are compatible $\iff$ They have compatible normal series.

Computability

Given two transitive groups. Recall our problem is to determine whether they are compatible.

Algorithm

- Input two groups. Take $n = 1$.
- Check all groups of order n. Determine whether any of them are witness.
- If so, the two groups are compatible. If not, take $n = n + 1$ and return to step 2.

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Difficulty on computation: If compatible, algorithm will stop in finite time. But what if incompatible?

Question: $\exists$? an algorithm which can determine whether two groups are compatible.

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Regular case:

- Regular compatible groups have compatible subnormal series. What about the converse?
- Are groups of the same square-free order mutually compatible?
- Are all groups of the same prime-power order mutually compatible?
- Are A_4 and C_{12} (smallest pair with compatibility unknown) compatible?